Pricing of Equities with Embedded Optional Clausis : Study of PPR's proposal on Gucci

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Abstract

When a financial asset is attached with an option whose underlying is the financial asset itself, both the price and its dynamics are modified. This article is based on a concrete example that occured in September 2001 when PPR offered Gucci's shareholders a guarantee on the price of the Gucci shares in 2 years and a half. We have modeled the mechanism of return to equilibrium after this offer, and we could then predict the price change and the deformationof the local volatility function of the Gucci shares. The observed shift on the price gives information about the implied default probability of PPR relative to Gucci's shareholders.

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1 Introduction

From the viewpoint of corporate finance operations, the years 1999 and 2000 have been very lively. Some of these operations have been in the news for months ; after two years of a mediatic and financial battle between Pinaud-Printemps-Redoute (PPR) and LVMH for buying the italian brand Gucci, PPR has been allowed to buy 5.9 millions dollars of shares Gucci provided that the other shareholders were offered the following contract : on april 30th, 2004, PPR undertakes to buy the last shares at a minimum price of 101.5 \$. This contract is a put option offered to the Gucci shareholders, but the underlying asset is the Gucci stock itself. How does this contract impact the price of the Gucci stock ? How is the dynamics of the stock price modified ?

At first sight, it is obvious that the new Gucci stock is a hybrid asset, a kind of convertible stock at a fixed date. The stock is now made of the stock Gucci and a put option on the stock itself since the shareholders have the right to sell the stock at a fixed strike, at the maturity date of the contract. We see that the structure of the asset changes deeply, and in particular, its dynamics changes because the price cannot be less that the price of a zero-coupon bond, with a nominal equal to the strike of the contract, and a maturity equal to the exercise date. However, the pricing remains a problem because the underlying asset and the put option refer to each other.

We can tackle this problem by considering how the market returns to equilibrium after PPR's offer. The way the market reaches the equilibrium is a complex problem (Smale 1998), and we have choosen a very simple framework to solve it here. First, we assume that the price dynamics of the Gucci stock is a continuous time lognormal diffusion before PPR's offer. We also assume that just after the offer, the price adjustments, that are proportional to the excess demand on the Gucci shares, are proportional to the price of the put. This assumption is realistic because the price of the put represents the market value of the offer. If the put has a non-zero value, this means that the price of the share is underestimated relative to the market value of the offer. In this framework, we are able to show that the risk neutral equilibrium dynamics of the Gucci stock remains lognormal with a local volatility function. As and when we approach equilibrium, we are able to compute how the volatility function is distorted. We show that the volatility function is equal to zero under a threshold : the price of the Gucci stock cannot go down through this threshold. This threshold is exactly equal to the price of the zero coupon bond with nominal equal to the discounted strike of the put option. The solution we obtain is thus very satisfactory because it creates some unreachable regions for the stock price (through zero volatility regions for instance), and we expect that the dynamics of the Gucci stock price is very different according to the level of the strike of PPR's offer. If the offer is "in the money", the Gucci stock behaves as a zero coupon bond and has a volatility close to 0 ; if the offer is "out of the money", the Gucci stock behaves as the former Gucci stock, and the offer is worth nothing.

Another aspect of the offer, is that the exercise of the put option is also conditional on the existance of PPR's offer at the maturity. If PPR withdraws its offer before the maturity, the equations driving the return to equilibrium are modified. By introducing the probability that PPR is going to default relative to Gucci's shareholders, we are able to calculate the implicit value of this parameter from the observed market changes at the moment of the offer. This parameter, as we shall see, can also be interpreted as a risk aversion parameter of the investors.

The plan of this paper is as follows : in section 2, we recall some results about local volatility models. Section 3 is devoted to the pricing of stocks that include options such as the Gucci stock. This method leads both to the price of the stock and to its risk neutral dynamics. In section 4, we introduce the default risk on the embedded put option. This leads to the implicit degree of confidence of the investors relative to the PPR offer. Finaly, we conclude in section 5.

2 Local volatility models

In the Black-Scholes model, the structure of the volatility is flat, for any value of the strike and of the maturity. If this model were correct, the implicit volatilities would be the same for all the options. Empirically, this is not the case since there exists a structure by strike and by maturity of the volatilities. Dupire (Dupire 1994,1997) has shown that it was possible to find a risk-neutral dynamics for the underlying asset coherent with this volatility structure, by introducing a deterministic function $\sigma(S,t)$, called local volatility function. The risj-neutral dynamics of the underlying asset in this model is given by :

$$\frac{dS_t}{S_t} = r\ dt + \sigma(S_t,t)\,dW_t \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard brownian motion. The differential stochastic equation (1) has a unique strong solution provided that the volatility function is lipschitz relative to the variable S, uniformly in t (Protter 1990).

This constraint is quite strong ; in particular, the volatility function must be continuous and its first derivative must be bounded, uniformly in t. If these conditions are satisfied, the price $C(S,t)$ of a european option with maturity T and payoff function $g(S_T)$ is the unique solution of the following partial derivatives equation :

$$\begin{cases} C_t + \frac{1}{2}S^2\sigma^2(S,t)C_{SS} + rSC_S = rC \\ C(S,T) = g(S) \end{cases} \quad (2)$$

Let us give a few properties of this kind of dynamics, especially when the volatility is equal to zero under a given threshold. In what follows, we consider a model with a volatility function of the form $\sigma(S,t) = \sigma(S)$ described in figure 1. This function satisfies all the required regularity conditions and is equal to zero under a fixed threshold (here 90).

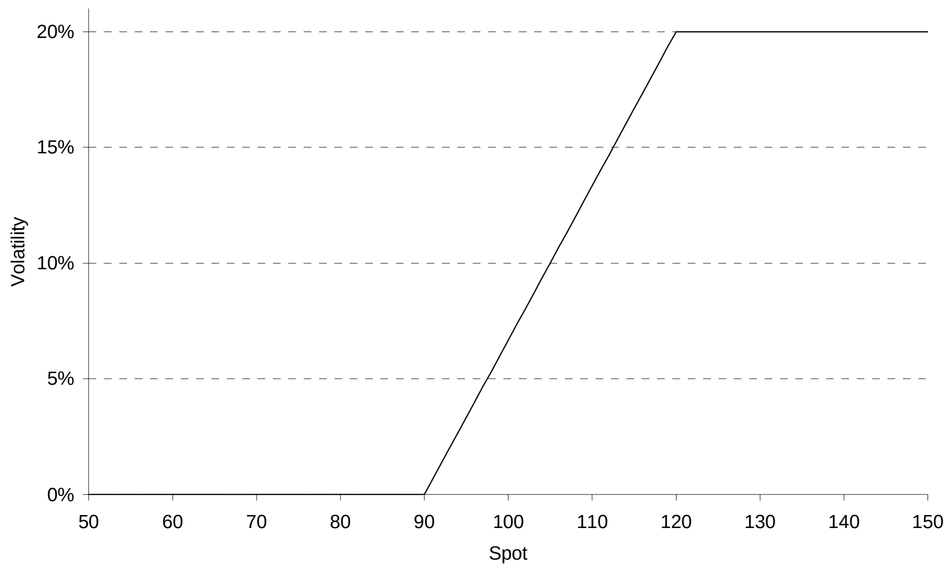

Figure 1 : An exemple of volatility function

An interesting property of an asset with such a volatility function is that if the initial value of the share is above 90, then it is never going to cross the threshold. Conversely, if the initial value of the share is under 90, then the price dynamics is deterministic as long as the threshold of 90 is not crossed. In order to illustrate this, we compute the price of a put option with maturity 1 year, strike 115 and a risk free interest rate of 5%. To this end, we solve equation (2) with finite differences method (Brennan 1978). When the volatility is zero, this method does not converge, but we know the the value of the option at each node of the grid because the dynamics of the underlying is deterministic in that case.

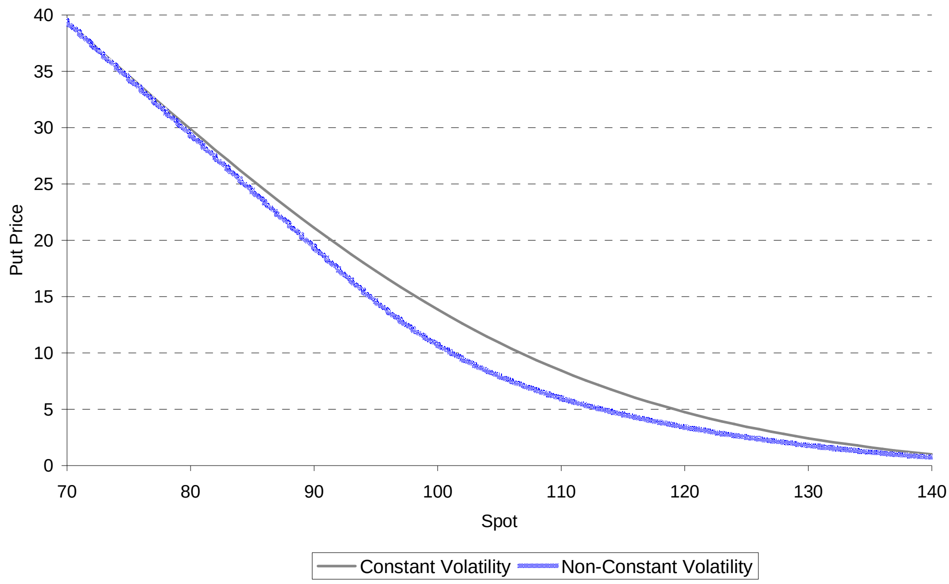

Figure 2 : Put price depending on the spot value and on the volatility function

We see on figure 2 that the curve (blue line) of the put option price in the case of a local volatility function is under the curve in the case of a 20% flat volatility function. This is what we expected because the put option is vega positive. We also see that when the spot price is inferior to the discounted value of the strike, then, the dynamics of the underlying asset is going to be deterministic until the end of the contract, and then, the put option price is a linear function of the underlying price. On the other hand, if the spot price is above the discounted value of the strike, then the dynamics of the underlying is going to be stochastic all the option life long : the curve deviates then from the linear curve. We also remark that, if the put strike is inferior to the discounted value of the 90, the put price is equal to 0 in the region where the spot is superior to 90 discounted because the probability that the put becomes in the money is equal to 0 at all date. This elementary remark is essential for what follows.

3 Self-coherence equations of the model

The starting point of the following analysis is that PPR's offer has no default risk. The general case is explored later. Before the offer, the Gucci stock is a usual stock, that we assume without dividends, and whose price follows a lognormal continuous time dynamics. We make the assumption of a perfect market and assume that the Gucci stock dynamics is given by the following SDE under the the risk-neutral measure :

$$\frac{dS_t}{S_t} = r\ dt + \sigma\ dW_t \tag{3}$$

where r is the riskless interest rate and σ the flat volatility of the Gucci stock. After the offer, Gucci's shareholders own, additional to their shares, put options on the stock Gucci with strike 101.5 \$ and maturity two years and a half. This option of course have a market value and have been offered to the shareholders. Hence, the Gucci stock has an embedded option and is replaced by a new asset including this option. This option changes the price of the stock, but also its dynamics until the maturity of the embedded option. Let us call $t = 0$ the date when PPR offers Gucci's shareholders the put option. The price process of the stock is called $(A_t)_{t \geq 0^+}$ and is different of the process $(S_t)_{t \leq 0^-}$ that was describing the dynamics of the Gucci stock before the offer. In this section, we are going to answer the following two questions :

1. At the instant of the offer, Gucci's stock price shifts from S_0 to A_0. What is the value of A_0 as a function of S_0 ?

2. What is the dynamics of Gucci's stock price after the offer ?

More precisely, we are going to show that if the dynamics of Gucci's stock price before the offer is a lognormal diffusion, then the dynamics after the offer is a local voltility model. We show that the risk neutral dynamics of Gucci's shares after the offer is the solution of the following SDE :

$$\frac{dA_t}{A_t} = r\ dt + \sigma(A_t, t)\ dW_t \tag{4}$$

After the offer, the price of the put option on the new Gucci shares have to be equal to zero since this put is now embedded in the share itself. If this is not the case, we can make an arbitrage by selling the stock and borrowing an amount equal to the strike of the option. It is easy to show that there are an infinity of local volatility functions $\sigma(A,t)$ that satisfy the constraint of a put price equal to zero : in particular, all the functions $\sigma(A,t) = f(A)1_{\{A \geq Ke^{-r(T-t)}\}}$, C^1 in A, make the put price equal to zero almost surely for all $t > 0$.

Let us assume that the financial market is complete and efficient, and that the arbitrages disappear instantaneously. We also assume that the dynamics of the Gucci shares is given by Eq.(3) before the offer for all the market participants. Similarly to the Grossman-Stiglitz model (Grossman 1980), we assume that the price variation between two transactions of the Gucci shares is proportional to the excess of demand. We call S^n the price of the Gucci's shares at the n-th transaction after the offer and we have :

$$S^{n+1} - S^n \propto XD(n) \tag{5}$$

where $XD(n)$ is the excess demand on the Gucci shares after the n-th transaction after the offer. If after the n-th transaction, the put option still has a positive market value, this means that the PPR offer is still attractive and the investors may be interested in buying the shares at price S^n. It is then very reasonable to assume that the excess demand is proportional to the put price :

$$S^{n+1} - S^n = \frac{P(S^n)}{\lambda} \tag{6}$$

The equilibrium of the market is realised when the put price is equal to zero. If we assume that the market participants price the stock Gucci one after the other, we see that the price process of the Gucci shares after the offer is the limit of the sequence of processes :

$$\begin{cases} S^{n+1} = S^n + \frac{P(S^n)}{\lambda} \\ S^0 = S \end{cases} \tag{7}$$

Up to now, when did not give any details on the pricing function of the put, but it is clear that the volatility function associated to the process $\left(S^{n+1}\right)_{t\geq 0}$ can be deduced recursively from the volatility function of the process $\left(S^n\right)_{t\geq 0}$. We show in the appendix :

$$\sigma\left(S^{n+1}\right) = \sigma\left(S^n\right)\left(1 + \Delta_P(S^n)/\lambda\right)\frac{S^n}{S^{n+1}} \tag{8}$$

where the function $\Delta_P(.)$ is the delta of the put. The parameter λ has several interpretations. In the Grossman-Stiglitz (Grossman 1980) model, the link between the price variation and the excess demand is the market depth : it is the size of a demand that increases the price of one unit per unit of time. However, in our model, we assume that the market is perfect and that the transactions that permit to reach the equilibrium take place in an extremely short period of time. Here, the market depth is infinite (since the equilibrium is reached instantaneously) but the equilibrium equation is very similar to Grossman-Stiglitz equation provided that we replace the time variable by a number of transactions variable.

If we assume that the market participants are not risk averse, then they consider that there is no default risk on PPR, and the first transaction is going to take place at the price $S^1 = S^0 + P(S^0)$. In this case, $1/\lambda = 1$, and we notice that the volatility function is non zero above a threshold equal to the discounted value of the strike. We thus have a solution that satisfies the constraints of the problem and in particular, the price of the put on the new share Gucci is equal to 0. This means that the equilibrium has been reached after the first transaction. The price of the new stock Gucci as a function of the price of the old stock Gucci is given in figure 3 :

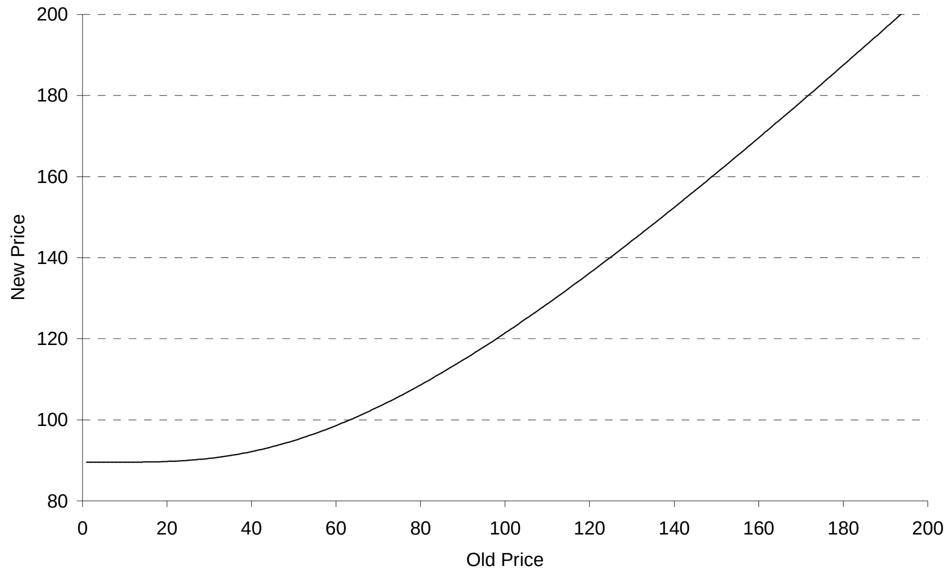

Figure 3 : New price of Gucci share after the announcing of PPR's offer

For the calculations, we have taken a riskless interest rate of 5% and an initial volatility of 45%. We also obtain the volatility function of the new stock Gucci :

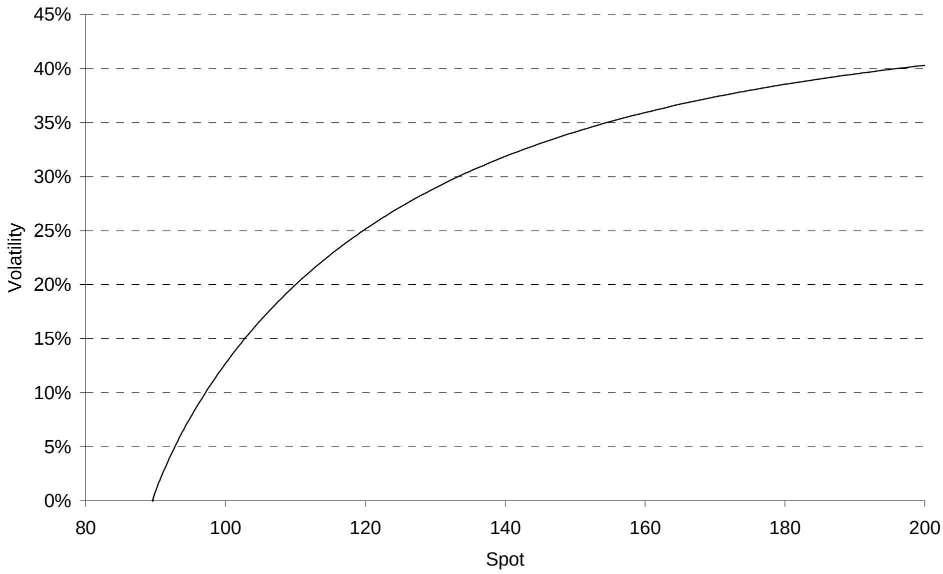

Figure 4 : New volatility function of Gucci equity

The offer increases the price of the Gucci stock. This new price is above the price of the zero-coupon bond with nominal equal to the strike of the put option offered by PPR. In the limit where the option is in the money, the price of the stock Gucci is equal to the discounted value of the strike offerd by PPR. In the case where the offer is out of the money, the price of the stock does not change.

The volatility function has a threshold exactly at the discounted value of the strike ; this means that the stock price cannot go down through this threshold. On the other hand, when the price of the Gucci stock goes far above the threshold, the offer becomes out of the money, and the volatility of the new stock Gucci goes closer and closer to the volatility of the old stock Gucci.

The formalism with discrete sequences of processes is not efficient when the coefficient $1/\lambda$ is not equal to 1 because, in this case, an infinite number of iterations are required to reach the equilibrium, and the numerical calculations in equation (7) are very complex. We propose here a formal limit where the number of transactions after the offer is a continuous variable that we call x. Let $(A_t(x))_{t\geq 0}$, the family of processes that are going to enter the self-coherent equations ; These are the price processes of the stock Gucci at transaction x. The initial condition is $A_0(0) = S_0$, and the equilibrium price process of the stock Gucci after the PPR offer is $(A_t(\infty))_{t\geq 0}$. In what follows, we are going to drop the time index t for sake of simplicity. In the case of non risk averse agents, the condition $1/\lambda = 1$ is replaced by $1/\lambda = dx$. We recover the discrete case for $dx = 1$. In the continuous limit, equation (6) becomes :

$$\frac{\partial A}{\partial x} = P\left[A(x), \sigma(., x)\right] \tag{9}$$

On the other hand, if we assume that the risk neutral dynamics of the process $(A(x))$ is given by the SDE (4), then we show that the process $(A(x + dx))$ is still a diffusion process, and Itô's lemma leads to the volatility function of this process through the self coherent equation :

$$\frac{\partial \sigma}{\partial x}(Y, t, x) = \left(\begin{array}{c} \left[\frac{\partial P}{\partial Y}\left[Y, t, \sigma(., ., x)\right] - \frac{P[Y,t,\sigma(.,.,x)]}{Y}\right] \sigma(Y, t, x) \\ -P\left[Y, t, \sigma(., ., x)\right] \frac{\partial \sigma}{\partial Y}(Y, t, x) \end{array} \right) \tag{10}$$

The proof is given in the appendix.

We have solve equations (9) and (10) numerically ; the pricing of the put thanks to Dupire's model, is described in section (2). The following figure gives the price of the new stock Gucci as a function of the price of the old stock Gucci for $x = 5$, compared to to the same function computed in the discrete framework

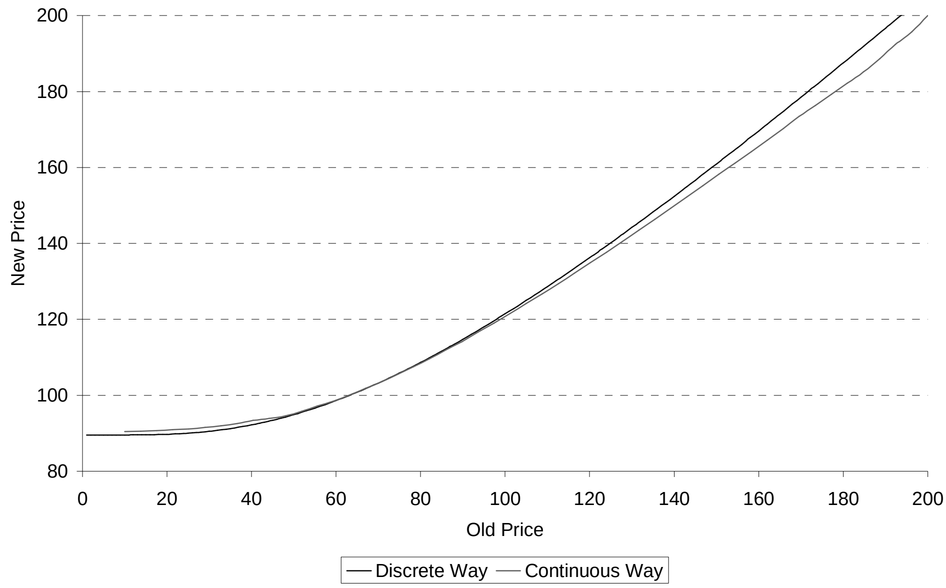

Figure 5 : Comparaison of the new price founded by the 2 ways

Despite of numerical problems, we see that the curves are very close to each other : both methods converge to the same solution.

4 Default risk of PPR

The analysis of section 3 starts from the fact that PPR is not going to default on the offer. This point is highly unrealistic and we assume that PPR can retire its offer at a random date that follows a Poisson law with parameter μ. At date t, the probability that PPR retires its offer before the maturity is :

$$p(t,T) = 1 - e^{-\mu(T-t)} \tag{11}$$

Equations (9) and (10) are slightly modified by this perturbation :

$$\begin{cases} \frac{\partial A}{\partial x} = (1 - p(t,T))\, P\left[A(x), \sigma(.,x)\right] \\ \frac{\partial \sigma}{\partial x}(Y,t,x) = (1 - p(t,T)) \begin{pmatrix} \left[\frac{\partial P}{\partial Y}\left[Y,t,\sigma(.,.,x)\right] - \frac{P[Y,t,\sigma(.,.,x)]}{Y}\right] \sigma(Y,t,x) \\ -P\left[Y,t,\sigma(.,.,x)\right] \frac{\partial \sigma}{\partial Y}(Y,t,x) \end{pmatrix} \end{cases} \tag{12}$$

The continuous limit is here necessary to solve the problem numerically. We draw hereafter the price curves as a function of the probability of occurence of the PPR offer

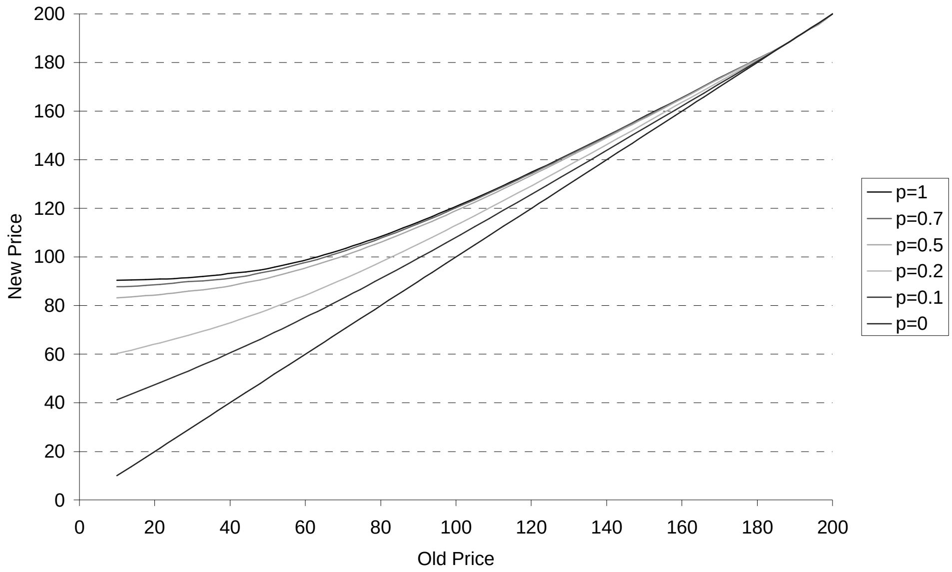

Figure 6 : New prices depending on the old prices and on the probability the offer will occure

Similarly, we obtain the volatility functions :

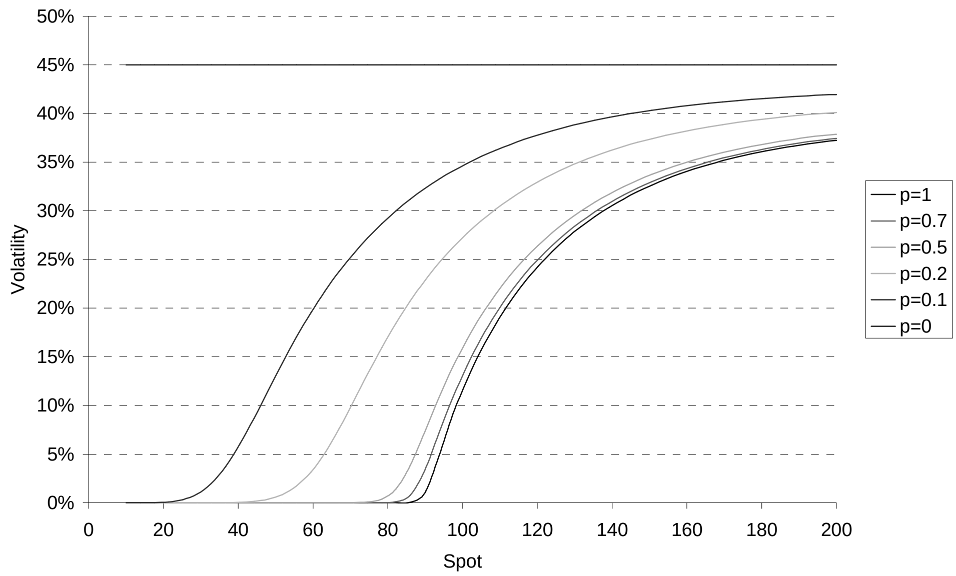

Figure 7 : Volatility functions of Gucci share

We notice that the prices are shifted a lot, even when the default probabilities are very low. These functions lead to a market implied default probability of PPR on its offer. In figure 8, we have the daily prices of the stock Gucci around the day of the offer. The volatility of Gucci is 45% and the interest rate is taken equal to the bond rate of PPR, that's to say 8%. On september 20th, 2001, day of the offer, the price of the stock Gucci jumped from 69.075 $ to 83.040 $.

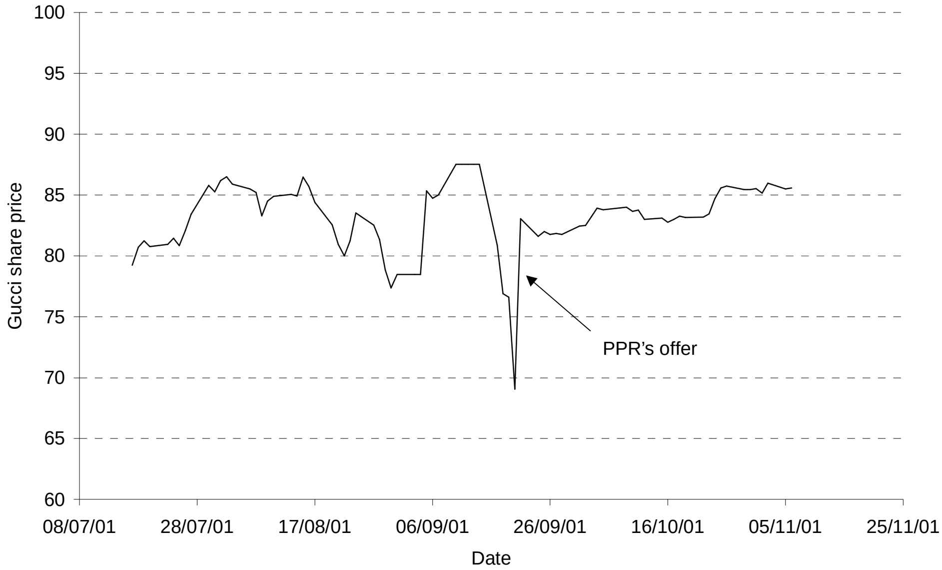

Figure 8 : Historical of Gucci share price around the Septbember 20th, 2001

In figure 9; we have the price of the new stock Gucci as a function of the occurence probability of the offer.

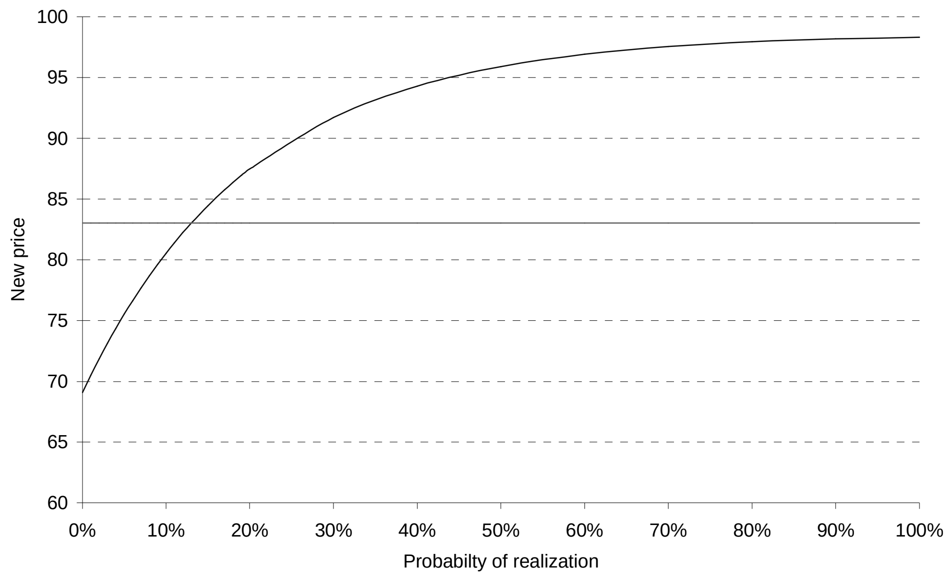

Figure 9 : New spot price depending on the realization probability

The jump of the price of the stock Gucci corresponds to an implied probability of occurence of 13.5%. The market does not seem to be convinced by PPR's offer or maybe expects other elements of information. We see that the equilibrium price is very sensitive to the occurence probability when it is low. This illustrates the impact of an announcement in the markets, even if it is only a non convincing rumor. On the light of figure 10, we estimate that a convincing offer corresponds to an offer that attracts 20-30% of the investors. For instance, if the occurence probability is only 30%, the price shift of the stock is 32%, whereas if the occurence probability is close to 100%, the price shift is 42%.

5 Post-offer behavior

In section 4, we supposed that the market reacted quickly to PPR's offer, that enabled us to fit our model parameters (in particular the probability of occurence). We can now check the adequacy of our model and parameters with the reality. Has the volatility decreased? Did the investors raise their confidence in the offer's probability?

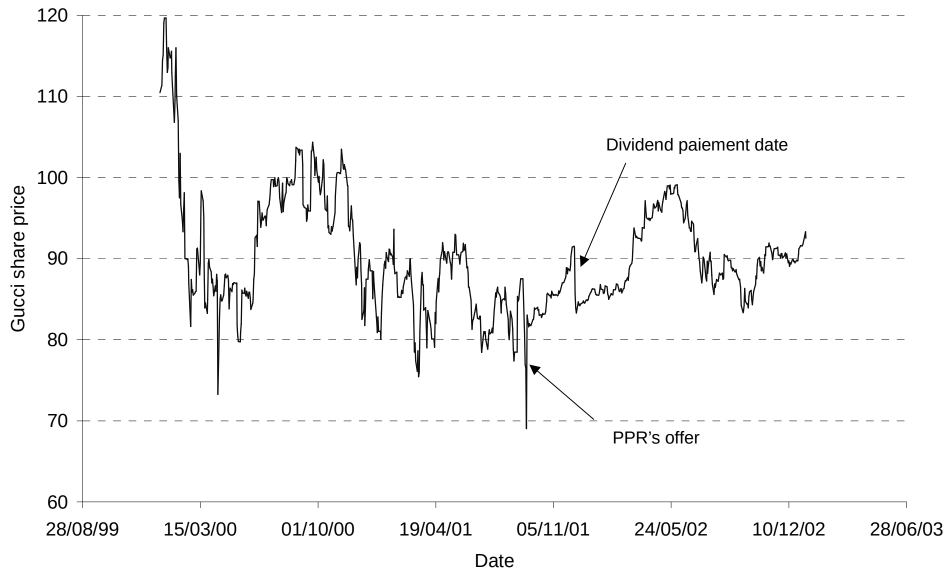

Figure 10 : Historical of Gucci share price since January, 2000

Let us first note that the drawdown on December 12th, 2001 is only caused by a dividend paiement. The average historical volatility was 41.5% before the offer and felt to 12.8% after, which is coherent with the prediction of the model that leads to a convenient qualitative behaviour for the Gucci stock dynamics.

6 Conclusion

In this paper, we have studied the impact of an offer, here an optional offer, on the stock price of the firm Gucci. Generally, the market is very sensitive to new informations and in particular in the case of an offer. The price of the assets may shift very violently and very quickly in order to include the market price of the offer. However, the equilibrium price is not easy to compute especially if the offer modifies the dynamics of the underlying stock.

We have developped a model that aims to compute the impact of an optional offer on the price and on the dynamics of the underlying stock.

We have shown that, in a simple diffusion model, the return to equilibrium of the market can be monitored by some self-coherence equations that lead to the equilibrium stock price and to the equilibrium dynamics.Through the PPR-Gucci example, we have shown that the equilibrium price and the volatility are both solutions of differential equations. However, the price shift observed the day of the offer does not correspond to the predicted shift. We thus introduce a risk aversion parameter of the investors and are able to implicit the the probability that PPR defaults relative to Gucci's shareholders. In the case of PPR's offer, this implied default probability is 86.5% meaning that the market is not really confident. The model also predicts a drop of Gucci's volatility after the offer, which has been observed since then.

Our model gives an interesting insight on how markets may react to new informations and also provide us with some implied data from stock prices shifts. This approach is an original step forward in the literature of price formation, at the crossroads of corporate finance, option theory and behavioural finance.

7 Appendix

The starting point of equation (10) is the application of Itô's lemma on each member of the equation (9) which can be written like

$$A(x+dx) = A(x) + dx\ P\left[A(x), \sigma(.,x)\right]$$

With Itô's lemma:

$$\begin{aligned} dA(x+dx) &= dA(x) + dxdP\left[A(x), \sigma(.,x)\right] \\ &= \text{drift } dt + \sigma\left(A(x), x\right) A(x) \left[1 + dx\frac{\partial P}{\partial A}\right] dW_t \\ &= \text{drift } dt + \sigma\left(A(x), x\right) \\ &\quad A(x+dx)\frac{A(x)}{A(x+dx)}\left[1 + dx\frac{\partial P}{\partial A}\right] dW_t \end{aligned}$$

We can obtain the local volatility of diffusion $(A(x+dx)_{t\geq 0})$ with a Taylor expansion in dx :

$$\sigma\left(A(x+dx), x+dx\right) = \sigma\left(A(x), x\right)\left[1 + dx\frac{\partial P}{\partial A} - dx\frac{P\left[A(x), \sigma(.,x)\right]}{A(x)}\right]$$

By using the property that

$$\begin{aligned}\sigma\left(A(x+dx),x+dx\right) &= \sigma\left(A(x)+dA(x)),x+dx\right)\\ &= \sigma\left(A(x)),x\right)+dA(x)\partial_A\sigma+dx\partial_x\sigma\end{aligned}$$

We finally obtain the equation (10)

$$\frac{\partial\sigma}{\partial x}(Y,t,x)=\left[\frac{\partial P}{\partial Y}\left[Y,t,\sigma(.,.,x)\right]-\frac{P\left[Y,t,\sigma(.,.,x)\right]}{Y}\right]\sigma(Y,t,x)-P\left[Y,t,\sigma(.,.,x)\right]\frac{\partial\sigma}{\partial Y}(Y,t,x)$$

The relation (8) is exactly the same as equation (10) but in a discrete framework. The proof is very similar.

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