# Model-independent ABS duration approximation formulas

Vivien BRUNEL*- Faïçal JRIBI

April 7, 2008

**Abstract**

Asset backed securities are sensitive to both credit risk and prepayment risk. We introduce a new approach for modeling prepayments, and we compute robust and accurate model-independent approximations of ABS duration and convexity. [1]

## 1 Introduction

Asset Backed Securities (ABS) are amortizing bonds which performance depend on a portfolio of reference assets. These securities are sensitive to several risks, mainly default risk and prepayment risk. Because the ABS market is more mature in the US, the litterature is essentially US, and is divided into two main streams. The first approach consists of econometric models of prepayment calibrated upon historical data ([4]). However, these models have failed, in the last decade, to catch events that did not have any historical precedent, especially during periods with a burst of prepayments as this was the case in the 90s. The second approach is based on option-theoretic models pioneered by Dunn and McConnel ([1, 2]), which link prepayment events to the optimal refinancing of a loan. These models are not currently used in the industry because they are difficult to calibrate from MBS market prices ([5]).

On the periphery of the academic litterature, many market participants are using simple actuarial models for pricing their books and assessing their risk. Traders and asset managers use standard pricing functions available from Bloomberg for instance, that have now become a market standard. This model is just discounting future cash-flows in the zero-default scenario under a Constant Prepayment Rate (CPR) assumption. The margin above the short term reference interest rate (for instance 3M Euribor) is called the Discount Margin (DM) and is the main return indicator that traders use for asset selection. Of course, such models are static models and they ignore the optionnality of prepayment to interest rate changes ; in particular, they do not catch the negative convexity region of MBS prices.

Most of ABS traders and asset managers are long in these securities and have to assess the risk of their books. In a world with static interest rates, the actuarial model is relevant because prepayment is no longer interest rate driven. Even if the yield of the ABS bond changes because of DM variation, the prepayment rate is not supposed to be correlated to DM changes. If we consider short time scales and if we are far from the optimal exercise of the prepayment option, the assumption of constant interest rates is also reasonable and the basic actuarial model leads to an interesting method for assessing sensitivities of ABS prices.

Contrary to what happens on traditional bond markets, ABS traders do not use the notion of convexity, mainly because ABS are considered to be a very stable asset class. Probably for the same reasons, the use of

*email : vivien.brunel@sgam.com

[1] We are grateful to Julien Tamine for his comments and interesting suggestions.

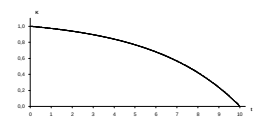
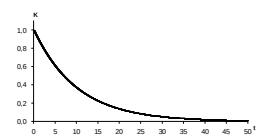
the WAL instead of the duration is very popular among traders and risk managers. Such market practices are questionnable. Risk assessment based upon incorrect assumptions can turn to be very dangerous when markets are getting more volatile. The recent crisis on ABS markets may change things.

The goal of this paper is to provide model-independent assessment for ABS risk. In section 2, we introduce a new formalism for prepayment. We then obtain general properties and approximations concerning ABS prices, WAL and sensitivities to risk factors. In section 3, we apply our methodology under the CPR assupmtion, and we show in particular that the approximations of the sensitivity obtained in this paper are accurate, contrary to the WAL.

# 2 Modelling framework

## 2.1 Amortizing asset

An amortizing asset is one that must be paid off over a specified time period, with regular payments of both principal and interest. Residential mortgage loans are perhaps the leading example of amortizing assets. We define the amortization profile from the outstanding principal balance of the asset over time. We denote by $\left(K_t^0\right)_{t\geq 0}$ the future outstanding principal balance at time $t$ scheduled at inception ($t=0$). Without loss of generality, we assume that the outstanding balance at time $t=0$ is equal to 1 (i.e. $K_0^0=1$) and that the asset is fully redeemed for large $t$ (i.e. $\lim\limits_{t\to\infty} K_t^0 = 0$). We show well known amortization schedules below :

| **Profile** | **Differential equation** | **Parameters** | **Graphic** |
|---|---|---|---|
| Installment loan | $\frac{dK_t^0}{dt} = r_M K_t^0 - x$ | $r_M$ is the mortgage interest rate and $x$ the constant payment rate |  |
| Fixed principal loan | $\frac{dK_t^0}{dt} = -\frac{1}{T}$ | $T$ is the asset maturity |  |
| Relative constant | $\frac{dK_t^0}{dt} = -k\cdot K_t^0$ | $k$ is the constant relative amortization rate |  |

The «Installment loan» is a loan which is repaid with a fixed number of equal-sized periodic payments. It is the most common way for amortizing an interest-bearing loan. Sometimes, the outstanding principal at maturity of the loan is non-zero: this is the «balloon loan». A balloon loan with a 100% outstanding principal amount at maturity is a «bullet loan». In a «Fixed principal loan», the principal portion of installments remains constant for the whole term of the loan. The third example appears when the borrower redeems a fixed percentage of the outstanding amount per time period. These examples illustrate the main different patterns we can obtain in an amortization schedule. For instance, the «Installment loan» schedule is concave, meaning that the amortization rate of the debt increases over time, contrary to the «Relative Constant» amortization profile. The linear profile decribes the situation in between.

The borrower may redeem its debt, either partially or totally, faster than scheduled, inducing an increase of the amortization rate. This is called prepayment. Under high prepayment scenarios, a concave theoretical schedule as the one of the «Installment loan» may be transformed into a convex schedule. The prepayment rate is generally random and we only know an estimate of the average prepayment rate at inception ($t = 0$) of the loan. The situation is the same for a pool of amortizing loans, that can be described by a theoretical amortization schedule (obtained by aggregating individual profiles) and by a prepayment scenario. The formalism that we are going to develop does not depend on the number of underlying loans in the pool.

We introduce the process $(Q_t)_{t\geqslant 0}$, which represents the percentage of the initial loan (or pool of loans) still outstanding at time $t$. The process $(Q_t)_{t\geq 0}$ can be either deterministic or stochastic, continuous or including jumps, and it may also include some dependency to interest rates. It is a positive decreasing process starting at time $t = 0$ from $Q_0 = 1$. We call $(K_t)_{t\geq 0}$ the resulting oustanding balance of the amortizing asset at time $t$, then $K_t = Q_t K_t^0$. More generally, if we consider a security backed by an amortizing asset, the resulting amortization schedule is more complex. For instance, for a mezzanine ABS with sequential amortization, attachment point A and detachment point D , we have:

$$K_t = \frac{max\big(0, min\big(D, Q_t K_t^0\big) - A\big)}{D - A}$$

From now on, $(K_t)_{t\geq 0}$ designates the amortization schedule of an ABS, $\left(K_t^0\right)_{t\geq 0}$ is the theoretical amortization schedule of the reference pool of assets and $(Q_t)_{t\geqslant 0}$ is its prepayment process. ABS traders call the quantity $K_t = f\left(Q_t K_t^0\right) \leq 1$ the «factor».

## 2.2 Prepayment and measure theory

As $-\int_0^T dK_t^0 = 1$ and $-\int_0^T dK_t = 1$, the theoretical and real principal redemptions generate two probability measures called $\mathbb{P}_0$ and $\mathbb{P}$ respectively. We consider a measurable function $A(t)$ with respect to $\mathbb{P}_0$ and $\mathbb{P}$. We introduce $\langle\,.\,\rangle$ and $\langle\,.\,\rangle_0$ the integration operators defined as:

$$\langle A\rangle_0 = -\int_0^\infty A(t)\, dK_t^0 \text{ and } \langle A\rangle = -E\left[\int_0^\infty A(t)\, dK_t\right]$$

In the definition of the bracket $\langle\,.\,\rangle$, the expectation is taken over all realizations of the prepayment processes$(Q_t)_{t\geqslant 0}$. The brackets stand for the expected value of the quantity $A(t)$ over the probability measure induced by the principal redemptions. Prepayments are changes in the timing of principal redemptions. Stated thus, introducing prepayments can be considered as changing this probability measure. Indeed, if we denote by $(F_t)_{t\geq 0}$ the Radon-Nikodym derivative of $\mathbb{P}_0$ with respect to $\mathbb{P}$, defined by $dK_t^0 = F_t \cdot dK_t$, then we get:

$$\langle A\rangle_0 = \langle A\, F_t\rangle$$

As we have $F_t \geq 0$, $\mathbb{P}_0$ is absolutely continuous with respect to $\mathbb{P}$. This formalism applies whatever the prepayment process, which can be either deterministic or stochastic. Within this formalism, we can easily write the usual quantities that characterize an ABS, namely Weighted Average Life (WAL) and price. We define the WAL by:

$$WAL = -E\left[\int_0^T t\, dK_t\right] = \langle t\rangle$$

The WAL has a clear interpretation in this framework. As mentionned in the introduction, we consider the simplest actuarial model, linking the price to the discount margin, just by discounting the future cash-flows

under the zero-default scenario at a risky rate. The instantaneous cash-flow at each date $t$ is the sum of principal payments $-dK_t$ and interest payments $y_0 K_t dt$, where $y_0 = r + s$ is the coupon rate paid by the security ($r$ is the reference risk free interest rate and $s$ is the premium paid by the security). The discount rate is equal to $y = r + DM$, where DM is called the Discount Margin. In orther words, the discount margin is the market spread of the security. The price is given by the following expression:

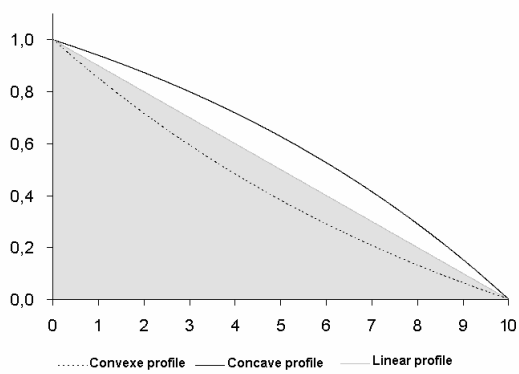
$$P = E\left[\int_0^T e^{-yt}[-dK_t + y_0 K_t dt]\right] \Rightarrow P - 1 = (y_0 - y) \cdot \frac{1 - \langle e^{-yt} \rangle}{y}$$

These formulas for WAL and price are completely general and do not depend on the prepayment model or on the amortization schedule of the asset. As we can see, they provide an implicit relationship between price and WAL, the intermediate state variable being the prepayment process.

## 2.3 General properties

### WAL and amortization schedule convexity

Linearity of the WAL is straightforward from its definition. Using integration by parts, we can express the WAL differently as $WAL = E\left[\int_0^T K_t dt\right]$. Thus, the WAL represents the area laying under the $E[K_t]$ curve (which is the expected amortization schedule). As illustrated below, if $E[K_t]$ is a convex function of $t$, then we have $WAL \leqslant \frac{T}{2}$ and if it is concave, then $WAL \geqslant \frac{T}{2}$ .

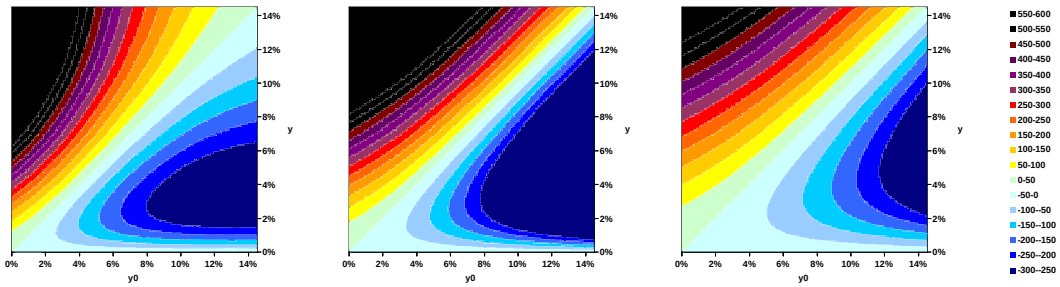



### Price bounds

Because of the convexity property of the exponential function, the price falls between two nontrivial bounds. We call $P_{bullet} = e^{-yWAL}\left(1 - \frac{y_0}{y}\right) + \frac{y_0}{y}$ the price of the bond having the same characteristics as the original bond except that is bullet with maturity equal to $WAL$. Form now on, for each amortizing asset, we call this bond the *associated bullet asset* and we denote by $P_{bullet}$ its price. Jensen's inequality ($\langle e^{-yt} \rangle \leq e^{-y\langle t \rangle}$) leads to the following bounds:

$$|P - 1| \leq |P_{bullet} - 1| = |y - y_0| \cdot \frac{1 - e^{-yWAL}}{y}$$

The graphics below illustrate the difference in bps between the price of an «Installment» amortizing asset and its *associated bullet asset* price for several prepayment levels.




$T = 30,\ \lambda = 0\%$ $\qquad$ $T = 30,\ \lambda = 10\%$ $\qquad$ $T = 30,\ \lambda = 20\%$

Figure 1: $(P - P_{bullet})$ in bps

### 2.4 Approximating sensitivity and convexity

As emphazised by Thomson in [6], the relative value between two ABS depends on the cash-flow dispersion. We show here that this is also the case for the yield sensitivity and convexity. Using the second-order Taylor series expansion (for small $y.t$ and $y_0.t$) of the price sensitivity to yield changes we obtain the approximation $\frac{\partial P}{\partial y} \sim -WAL + \frac{1}{2}(2 \cdot y - y_0)\langle t^2 \rangle$. On the other hand, we get from the second-order expansion of the price $\langle t^2 \rangle \sim \frac{2}{y}\left[WAL + \frac{P-1}{y-y_0}\right]$. This leads to the following general approximation of the price sensitivity to yield changes:

$$\frac{\partial P}{\partial y} \sim \frac{(2 \cdot y - y_0)(P-1) + (y-y_0)^2 WAL}{y\,(y-y_0)} \equiv \left(\frac{\partial P}{\partial y}\right)_{approx}$$

Unfortunately, this expression is singular at par ($y = y_0$) and the approximation no longer holds. In a similar way, we can approximate the price convexity to yield changes and we obtain:

$$\frac{\partial^2 P}{\partial y^2} \sim \langle t^2 \rangle \sim \frac{2}{y}\left[WAL + \frac{P-1}{y-y_0}\right] \equiv \left(\frac{\partial^2 P}{\partial y^2}\right)_{approx}$$

They are very interesting formulas because they are very accurate (see section 3) and have only global market data such as the price, WAL and yield as inputs. In particular, the approximations are independent from the underlying characteristic details of the the asset such as its amortization schedule, credit enhancement and tranche size.

## 3 Results in the CPR model

This section is devoted testing the approxiamation formulas in the constant prepayment rate (CPR) framework. In this case, the prepayment process is the exponential function $Q_t = e^{-\lambda t}$, where $\lambda$ is the prepayment rate. It is then easy from the theoretical amortization schedule $\left(K_t^0\right)_{t \geq 0}$ to compute the real amortization schedule of any structured product: $K_t = f\left(e^{-\lambda t} K_t^0\right)$.

### 3.1 Pass-through structure

In the case of a pass-through security, the cash-flows generated by the pool of reference assets are transfered to the security holders. The amortization schedule of the security is $K_t = e^{-\lambda t} K_t^0$, which is the solution of the following differential equation:

$$dK_t = Q_t dK_t^0 + \frac{dQ_t}{Q_t} K_t = e^{-\lambda t} dK_t^0 - \lambda K_t dt$$

This equation states that the principal amount redeemed between time $t$ and time $t + dt$ is the sum of the natural amortization of the asset (scheduled amortization) and prepayments (unscheduled amortization). As a function of the constant prepayment rate $\lambda$, the WAL writes $WAL(\lambda) = \int_0^{+\infty} e^{-\lambda t} K_t^0 dt$ and is the Laplace transform function of the amortization schedule $K_t^0$ with respect to $\lambda$. As we have $\frac{dWAL(\lambda)}{d\lambda} = -\left\langle t^2 \right\rangle /2$ and $\frac{d^2 WAL(\lambda)}{d\lambda^2} = \left\langle t^3 \right\rangle /3$, we conclude that $WAL(\lambda)$ is a decreasing and a convex function of $\lambda$.

Let us call $W(z) = \mathcal{L}_Z(K_t^0)$ the Laplace transform of the function $K_t^0$ at point $z$, then $WAL(\lambda) = W(\lambda)$ and the pass-through ABS price can be expressed as below:

$$P(\lambda,\, y) = 1 - (y - y_0) \cdot W(y + \lambda)$$

Besides, stated thus, we could easly prove that the price satisfies the following partial differential equation

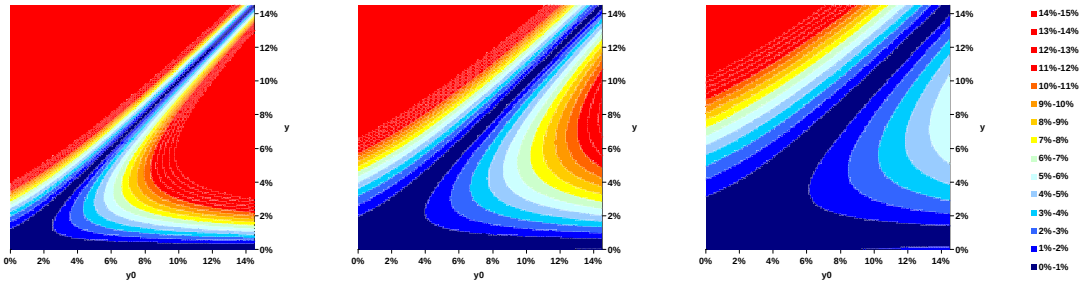
$$\frac{\partial P}{\partial \lambda} = \frac{\partial P}{\partial y} - \frac{P - 1}{y - y_0}$$

where $\frac{\partial P}{\partial \lambda}$ denotes the partial derivative of the price with respect to the prepayment ratio $\lambda$. If the prepayment rate increases, the WAL decreases and mechanically, the yield decreases because of the roll-down of the yield curve. If we call $S$ the slope of the yield curve, we can express the price sensitivity to $\lambda$, denoted by $S_\lambda$ as

$$S_\lambda = \frac{dP}{d\lambda} = \left(1 + S\frac{\partial WAL}{\partial \lambda}\right)\frac{\partial P}{\partial y} - \frac{P - 1}{y - y_0}$$

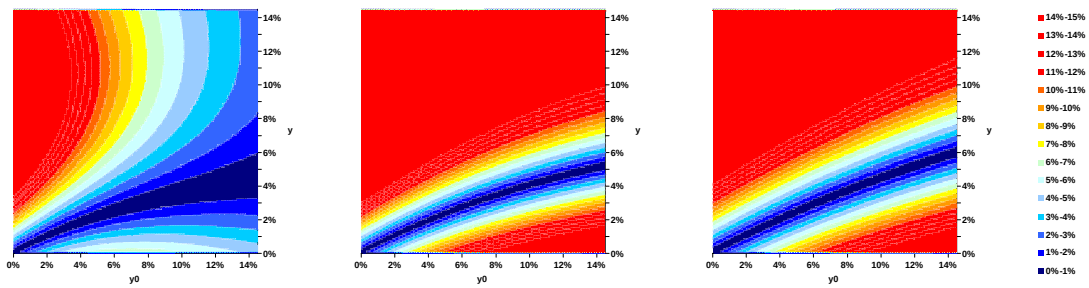
We can see for instance that when the ABS is at par, the sensitivity to the prepayment rate comes only from the roll-down of the ABS spread curve.

Concerning the sensitivity to the yield, we have three approximations at disposal, namely $-WAL$ which is extensively used by market participants and risk managers, $\left(\frac{\partial P_{bullet}}{\partial y}\right)$and $\left(\frac{\partial P}{\partial y}\right)_{approx}$, obtained in section 2. The graphics here below show the relative difference between these approximations and the exact value of the price sensitivity in the plane $(y_0, y)$.



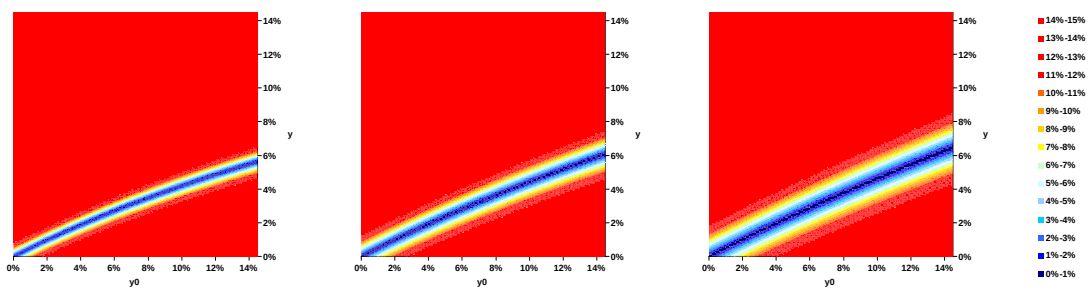

$T = 30,\ \lambda = 0\%$ $T = 30,\ \lambda = 10\%$ $T = 30,\ \lambda = 20\%$

Figure 2: Price sensitivity compared to $\frac{\partial P}{\partial y}_{approx}$




$T = 30,\ \lambda = 0\%$ $T = 30,\ \lambda = 10\%$ $T = 30,\ \lambda = 20\%$

Figure 3: Price sensitivity compared to $\frac{\partial}{\partial y} P_{bullet}$




$T = 30,\ \lambda = 0\%$ $T = 30,\ \lambda = 10\%$ $T = 30,\ \lambda = 20\%$

Figure 4: Price sensitivity compared to $-WAL$

The graphs of fig. 2, 3 and 4 lead to several comments. Firstly, the approximation of the sensivity by $-WAL$ is not accurate except for short term, high grade assets with low convexity (i.e. low yield), or in the particular case $2y - y_0 = 0$ in which the 2nd order convexity term in $\frac{\partial P}{\partial y} \sim -WAL + \frac{1}{2}(2 \cdot y - y_0)\langle t^2 \rangle$ vanishes. For large values of $y$ the WAL ignores discounting whereas for small values of $y$, the WAL ignores cash-flow dispersion. The second comment is that the approximations are better for low maturities or high prepayment rates (which is of course equivalent to low maturities) because the approximations are based

upon a Taylor series expansion in terms of $\langle (yt)^n \rangle$. The third comment is that around par ($y = y_0$), the quantity $\left(\frac{\partial P}{\partial y}\right)_{approx}$ is a good approximation of the sensitivity. The fourth comment is that when the yield to maturity of the asset decreases to 0, the approximation $\left(\frac{\partial P}{\partial y}\right)_{approx}$ is also very accurate because the impact of discounting is small.

### 3.2 Senior ABS tranches

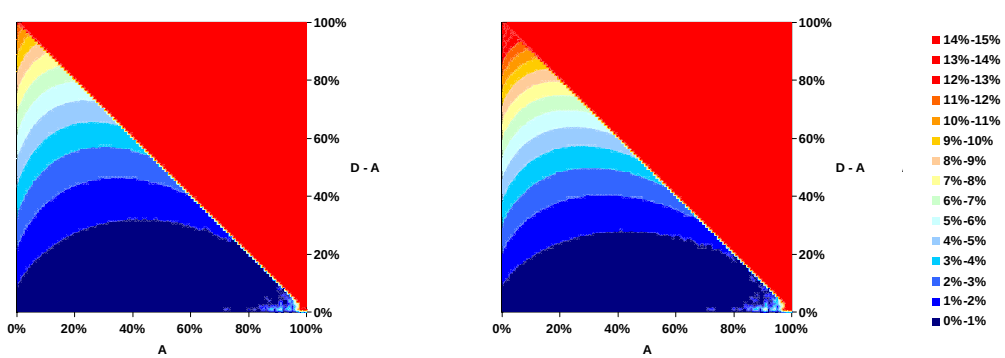
Additional concepts need to be detailed for sequential ABS modelling. Each tranche is defined by a detachment $D$ and an attachment point $A$ where $0 \leq A < D \leq 1$. Stated thus, the tranche is called «Senior» if $0 < A < D = 1$, «Mezzanine» if $0 < A < D < 1$ and «Junior» or «Equity» if $0 = A < D < 1$. In addition, starting from a principal balance of 1, the aggregate assets outstanding balance decreases over time because of redemptions. As long as it is higher than D, the detachment point of a given tranche, the latter's outstanding balance is still intact. From $D$ to $A$, the tranche investors receive all the assets' payments and the tranche outstanding balance decreases until it is paid off. The approximation of price sensitivity to yield change that we found in section 2, is even more efficient for senior tranches than pass-through securities. Indeed, senior tranches have shorter maturities and usually smaller coupons and yields thanks to the credit enhancement they benefit from subordinated tranches. However, the price sensitivity to the prepayment rate has a more complicated expression compared to the pass-through securities and requires numerical computation to be estimated.

### 3.3 Mezzanine ABS tranches

For thin mezzanine tranches, we expect that the approximation obtained in section 2 is not the most accurate. Indeed, in the case $D - A \to 0$, we obtain Infinitely Thin Tranches (ITT) that is a bullet exposure with maturity equal to $WAL$; we compute the price sensitivity directly:

$$\frac{\partial}{\partial y} P_{bullet} = -\frac{y_0}{y^2}\left(1 - e^{-yWAL}\right) - \frac{y - y_0}{y} \cdot WAL \cdot e^{-yWAL}$$

Fig. 5 illustrates the accuracy of both approximations of the mezzanine price sensitivity to yield for different tranche sizes $D - A$ and attachment points $A$. We considered an ABS which reference pool of assets has an "Installment loan" amortization schedule with a final maturity of 30 years and a CPR of 10% and values are calculated when the price is at par for two different values $y_0 = 5\%$ and $y_0 = 7\%$.




$y = y_0 = 5\%$ $\qquad$ $y = y_0 = 7\%$

Figure 5: Mezzanine price sensitivity approximation by associated bullet asset

The sensitivity approximation $\frac{\partial}{\partial y}P_{bullet}$ is robust for tranches up to 20% of thickness, and for almost all attachment points. This approximation is much better that the one obtained in section 2.

# 4 Conclusion

In this paper, we showed that the price of a secutity is equal to the discount factors weighted by the future cash-flows. As principal redemptions define a probability distrubution under the zero default scenario, the impact of prepayment is a chage of measure. This result is general and does not depend on the prepayment model. We obtain a very user-friendly formalism especially in the CPR model.

We find some accurate approximations of price sensitivity to the DM, of the convexity and of the sensitivity to the prepayment rate. We compared the sensitivity to the DM with the proxies generally used on the markets, namely the WAL or the sensitivity of the associated bullet. We showed that this proxy is not reliable, and we found an approximate model-independent formula that involves only on market data such as WAL, price and DM. This approximation is very accurate on a wide range of the parameter space.

A natural extension of this model would be a stochastic intensity based model for the prepayment rate. The issue for more complex models is calibration, but it could provide a relationship between prepayment volatility and ABS price volatility. In particular it could describe the proportion of the DM volatility that is explained by prepayment volatility.

# References


[1] Dunn, K. and McConnel, J., A comparison of alternative models for pricing GNMA Mortgage Backed Securities, Journal of Finance 36 (1981) 471-483.

[2] Dunn, K. and McConnel, J., Valuation of Mortgage Backed Securities, Journal of Finance 36 (1981) 599-617.

[3] Schönbucher, P., Credit derivatives pricing models, Wiley (2003).

[4] Schwartz, E. and Torous, E.S., Prepayment and the valuation of the Mortgage Backed Securities, Journal of Finance 44 (1989) 375-392.

[5] Tamine, J and Gaussel, N., Pricing Mortgage Backed Securities: from optimality to reality, working paper, SGAM AI (2003).

[6] Thomson, A., Evaluating amortizing ABS: a primer on static spread, in The handbook of fixed income securities, ed. Fabozzi, Mc Graw-Hill (2001).