Pricing credit derivatives with uncertain default probabilities

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Abstract

One main problem of credit models, as in stochastic volatility models for instance, is that the range of arbitrage prices of risky bonds and credit derivatives is generally very wide. In this article, we present a model for pricing options on the spread in an environment where the rating transition probabilities are uncertain parameters. The transition intensities are assumed to lie between two bounds which can be easily interpreted in the light of the rating agencies' transition matrices. These bounds are a confidence interval of the rating transition intensities. We show that the bounds of arbitrage prices are solutions of a non-linear partial differential equation. In particular, when using realistic values for the rating transition (default) probabilities, the arbitrage range of credit derivatives prices remains narrow.

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1. INTRODUCTION

Credit derivatives are derivative securities whose payoff is contingent to the credit quality of a given obligor. This credit quality is measured by the credit rating of the obligor or by the spread of his bonds over the yield of a similar default free bond. In this article, we focus on credit spread options.

From a theoretical point of view, a benchmark model as Black-Scholes' model for equities, is still lacking for credit derivatives. This is an obvious obstacle to the development of credit derivatives markets. Actually, the question is rather complex because, as we shall see, credit risk usually introduces incompleteness in the market since changes in the credit quality modify the dynamics of the risky assets but cannot be hedged away.

The way to tackle incomplete market problems is three-fold. We can choose a utility based method, as the one introduced by Davis ([9]) in order to price credit derivatives, but the main problem is that this method depends on the agents' preferences. Another approach is to select a criterion in order to choose one equivalent martingale measure out of the infinite set of equivalent martingale measures available ; for instance, Follmer and Sondermann ([12]) have proposed the criterion of minimization of the quadratic risk in order to select an equivalent martingale measure. The last approach is to find the range of prices within the arbitrage bounds for credit derivatives and to keep all the equivalent martingale measures in the calculations. This method leads to solve the super-replication problem which consists in finding the cheapest portfolio made of the underlying asset and the riskless asset whose terminal value is almost surely superior to the payoff of the option. The key point with any of these approaches is the duality existing between the hedging problem an the set of equivalent martingale measures.

The last approach is of course the most satisfactory because it is not based on a choice of a utility function or risk measure ; however, it generally gives a trivial range for derivatives prices ([7,10,20]). For instance, in the case of credit risk models, the range of prices of a risky bond is simply determined by all the possible dates of occurrence of the default : the lower bound is the price of the bond if ever the default is going to occur immediately, and the other bound is the price for a riskless bond (see [7]). Thus, as in usual option models in incomplete markets, the problem of super-replication for credit derivatives often leads to trivial arbitrage prices.

Here, we propose a new methodology in order to get non trivial arbitrage bounds on credit derivatives prices. The market is made of one riskless asset and one risky bond. As in Black-Scholes' or Vasicek's model ([4,22]), we specify a continuous time dynamics for the underlying asset (here it is the spread or, equivalently, the price of the risky bond), and we consider a European option written on this asset. We also assume that the rating of the issuer can change, and the probabilities of such changes are given by the rating transition matrices. In our model we assume that the spread of the risky bond follows an Ornstein-Uhlenbeck process with rating-dependent coefficients. The incompleteness of the model comes from the rating transitions that cannot be hedged away by trading on the only asset available in the market. The main idea of our model is to deal with a subset of the equivalent martingale measures only, as compared to bounded stochastic volatility models where we assume that the volatility lies between two extreme values. The bounded uncertain parameters are the intensities of rating transitions. The rating agencies give some statistics about the rating transition probabilities but for a given firm, the transition probabilities remain unknown. Indeed, the rating agencies make their statistics on very large samples of firms and do not catch the specific risk of each firm. Our methodology permits to deal with the specific risk of the firm through uncertain transition probabilities. As we shall see, this leads to consider the super-replication price of spread derivatives which are solutions of a non linear Partial Differential Equation (PDE) that gives non trivial arbitrage ranges for credit derivatives prices.

This article is organized as follows : in the second section, we make a short empirical study of the dynamics of the spreads according to the maturity of the contracts and to the rating of the issuer. In section 3 we describe the generating matrix formalism in order to model the rating transitions. Then, in section 4, we present the continuous time model and derive the non linear PDE that gives the arbitrage range of derivatives on the spread is described in section 5. These equations are solved numerically in section 6 in a simple three rating levels model. Section 7 concludes.

2. SPREAD DYNAMICS

From an econometric point of view, the process of the logarithm of the spread is often modeled by an Ornstein-Uhlenbeck process ([19]). However, this implies a positive value for the spreads. This is not always the case, for instance when we only consider the spread between high quality corporate rates and swap rates. In this article, we consider spreads over swap rates ; this sometimes leads to negative values for the spread. Thus we are going to assume that the spreads time series follow an autoregressive AR(1) process of the form :

$$X_{t+1} - X_t = a(R,T)\left[b(R,T) - X_t\right] + s(R,T)\varepsilon_t \tag{1}$$

This discrete dynamics involves three parameters : the parameter $a(R,T)$ is interpreted as the mean reverting speed, $b(R,T)$ is the long term equilibrium value of the spread and $s(R,T)$ is a volatility parameter of the spread. The variables R and T are the rating of the issuer and the maturity of the debt we are considering (we only consider here long term debts and credit qualities). The random variable ε_t is a gaussian white noise. Let us note that there is a term structure of the spreads and that the non arbitrage conditions would imply relations between these parameters in a continuous time model. Here, we only consider equation (1) from an econometric point of view.

In order to estimate these three parameters for each value of the rating and maturity, we have selected indexes of US industrial bonds built by Bloomberg. Each index corresponds to a given rating and Bloomberg has reconstructed a yield curve for each sector and rating. The data are daily index yields from 02/28/98 to 12/01/99; the spreads we have calculated are the difference between these yields and the corresponding US swap rates of the same maturity. Our results are reproduced in Table 1. They provide the maximum likelihood estimations of the parameters $a(R,T)$, $b(R,T)$ and $s(R,T)$, and the 90 % confidence intervals of these estimates.

Mean (bp)	2Y	5Y	10Y	20Y	30Y
AAA	-5,14	-8,66	-15,25	10,63	13,17
AA	-0,38	-5,12	-9,12	16,62	20,41
AA-	4,97	2,23	-2,55	24,71	30,46
A+	12,36	12,62	6,95	36,79	42,04
A-	31,97	37,19	36,62	62,48	68,64
BBB+	42,89	53,33	52,53	76,54	86,75
BBB	54,56	63,37	64,55	93,81	100,23
BBB-	73,36	89,26	90,10	115,22	122,71

Confidence	2Y	5Y	10Y	20Y	30Y
AAA	38,80	33,37	32,14	66,72	86,47
AA	39,39	35,45	41,95	66,44	106,13
AA-	51,97	56,99	43,38	63,75	56,28
A+	86,66	93,64	55,56	81,12	88,78
A-	214,08	169,37	88,96	189,98	295,73
BBB+	286,89	273,72	177,30	238,13	382,88
BBB	339,38	276,79	238,93	376,96	532,22
BBB-	561,95	496,76	296,40	528,94	690,69

Speed	2Y	5Y	10Y	20Y	30Y
AAA	0,16	0,18	0,20	0,08	0,06
AA	0,15	0,17	0,15	0,08	0,04
AA-	0,12	0,11	0,14	0,09	0,08
A+	0,07	0,07	0,11	0,06	0,05
A-	0,03	0,04	0,07	0,03	0,02
BBB+	0,02	0,02	0,04	0,02	0,01
BBB	0,02	0,02	0,03	0,02	0,01
BBB-	0,01	0,01	0,02	0,01	0,01

Confidence	2Y	5Y	10Y	20Y	30Y
AAA	0,90	0,96	0,98	0,66	0,56
AA	0,87	0,93	0,87	0,65	0,49
AA-	0,80	0,75	0,87	0,67	0,67
A+	0,61	0,60	0,78	0,59	0,52
A-	0,42	0,46	0,63	0,38	0,30
BBB+	0,35	0,37	0,46	0,35	0,27
BBB	0,33	0,37	0,41	0,28	0,23
BBB-	0,26	0,28	0,36	0,24	0,21

Volatility (bp)	2Y	5Y	10Y	20Y	30Y
AAA	0,0367	0,0365	0,0379	0,0326	0,0289
AA	0,0348	0,0364	0,0371	0,0314	0,0273
AA-	0,0381	0,0362	0,0366	0,0328	0,0279
A+	0,0363	0,0374	0,0370	0,0313	0,0258
A-	0,0412	0,0403	0,0389	0,0312	0,0282
BBB+	0,0394	0,0399	0,0395	0,0339	0,0325
BBB	0,0408	0,0405	0,0428	0,0342	0,0323
BBB-	0,0382	0,0410	0,0423	0,0352	0,0340

Confidence	2Y	5Y	10Y	20Y	30Y
AAA	0,0306	0,0304	0,0315	0,0272	0,0240
AA	0,0289	0,0303	0,0308	0,0261	0,0227
AA-	0,0317	0,0302	0,0305	0,0273	0,0232
A+	0,0302	0,0312	0,0308	0,0261	0,0215
A-	0,0344	0,0336	0,0324	0,0259	0,0235
BBB+	0,0329	0,0333	0,0329	0,0281	0,0271
BBB	0,0338	0,0337	0,0357	0,0283	0,0269
BBB-	0,0317	0,0341	0,0351	0,0293	0,0283

Table 1 : Mean value, mean reverting speed and volatility parameters of credit spreads (left), and the corresponding 90% confidence interval (right).

These results provide interesting insights about the dynamics of credit spreads. First, the dynamics of credit spreads is mean reverting because of the positivity of the mean reverting speed of the process : for any value of the rating and of the maturity, the coefficient $a(R,T)$ is positive (see Table 1, 3rd and 4th tables). Moreover, as shown in Longstaff and Schwartz ([19]), the mean reverting speed of credit spreads decreases for lower-rated debts and also decreases with maturity.

The mean of the credit spreads is also a parameter of interest. In Table 1, we show that $b(R,T)$ is clearly increasing with the rating. This behavior is of course intuitively correct : for a lower rated debt, we expect a higher return.

Another conclusion of our empirical study is that credit spread volatility parameters increase as the debt quality decreases (see Table 1, 5th and 6th tables). Here again, our results are in agreement with the results obtained by Longstaff and Schwartz in [19]. Table 1 (5th and 6th tables) details the volatility parameter of the spread as a function of the rating and the maturity.

As we can see directly in the time series themselves, the spread process has jumps, and the spread variations are not normally distributed. The confidence intervals for the parameters confirm this affirmation : in Table 1, we have computed the width of the 90 % confidence intervals for each parameter. We observe that for the mean reverting parameter and for the long term mean value of the spread, the width of the confidence intervals are much larger than the parameters themselves. For the volatility parameter, the estimation is much better.

This analysis clearly shows that the spread process is far from a AR(1) process, even if the estimated parameters look friendly. However, in our continuous time model of section 4, we shall choose an Ornstein-Ulhenbeck process for the spread dynamics in order to get a tractable model. Before, this let us introduce the transition matrices formalism that models the rating changes.

3. TRANSITION MATRICES

In order to mathematically construct a coherent model for rating transitions, we consider the Markov chains formalism ([15,18]). The rating process is a jump process that takes its values in a finite set of integers. We assume that we have D levels of rating for the risky issuer, from 1=AAA to D=default and assume that the transition probability from level i to level j is proportional to the time interval :

$$P\left[R_{t+\Delta t} = j \middle| R_t = i\right] = b_{i,j}.\Delta t \tag{2}$$

Rating agencies like Standard and Poors or Moody's give a one year matrix transition. Standard & Poor's ([20]) one year rating transition matrix (april 1996) is given in Table 2.

Initial rating	Rating transition probability after one year (%)							
	AAA	AA	A	BBB	BB	B	CCC	D
AAA	90,81	8,33	0,68	0,06	0,12	0	0	0
AA	0,70	90,65	7,79	0,64	0,06	0,14	0,02	0
A	0,09	2,27	91,05	5,52	0,74	0,26	0,01	0,06
BBB	0,02	0,33	5,95	86,93	5,30	1,17	0,12	0,18
BB	0,03	0,14	0,67	7,73	80,53	8,84	1,00	1,06
B	0	0,11	0,24	0,43	6,48	83,46	4,07	5,20
CCC	0,22	0	0,22	1,30	2,38	11,24	64,86	19,79
D	0	0	0	0	0	0	0	100

Table 2 : One year rating transition probabilities (source : Moody's).

Of course, the transition matrix over a given period depends on the period length. If we assume a stationary property of the transition matrices (*i.e.* a matrix transition over a period does not depend on the date at which we consider the transition matrix), then we can easily build a transition matrix over any period from the one year matrices given by the rating institutes. Let us call $P(\Delta t)$ the transition matrix between *t* and $t+\Delta t$. We develop this matrix around the identity matrix up to first order in Δt :

$$P(\Delta t) \approx I + A.\Delta t \tag{3}$$

where *I* is the $D \times D$ identity matrix. Then, the transition matrix between time *t* and time *t+s* writes :

$$P(s) = e^{s.A} \tag{4}$$

Initial rating	Rating transition generating matrix (%)							
	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-9,68	9,18	0,35	0,02	0,14	0	0	0
AA	0,77	-9,96	8,57	0,45	0,01	0,14	0,02	0
A	0,09	2,49	-9,69	6,18	0,66	0,22	0,00	0,05
BBB	0,02	0,28	6,67	-14,50	6,29	1,03	0,09	0,12
BB	0,03	0,13	0,45	9,24	-22,40	10,70	1,07	0,77
B	0	0,12	0,24	0,10	7,86	-18,88	5,49	5,07
CCC	0,29	0	0,20	1,57	2,62	15,14	-43,77	24,00
D	0	0	0	0	0	0	0	0

Table 3 : Generating matrix.

Matrix A is called the generating matrix whose properties are described in [1,14]. It is of course possible to compute the generating matrix from the matrix provided by the rating agencies just by taking the logarithm of the transition matrix in a diagonal basis and by coming back in the original basis (the generating matrix is a stochastic matrix with a dominant diagonal, and thus is diagonalizable). The main property of the generating matrix is that the sum of the coefficients of a row of the matrix is equal to 0, and the only negative coefficients are the diagonal coefficients.

The main objection to this kind of model is that historical data are not consistent with the Markov property of the transition matrices autocorrelations of transitions and defaults. Let us take an example. In the above framework, the transition matrix corresponding to a ten-years maturity is equal to the one year transition matrix to the power ten. We have computed from Moody's data the default probability over a ten years period thanks to the one year transition matrix, and we have compared the results with the historical ten years default probabilities. Table 4 summarizes the results :

	1 Yr	10 Yrs (matrix)	10 Yrs (historical)
Aaa	0	0,3%	0,7%
Aa	0	1,1%	0,9%
A	0	4,0%	2,0%
Baa	0,2%	11,3%	5,0%
Ba	1,8%	34,3%	19,5%
B	8,3%	65,3%	40,0%

Table 4 : Ten years default probabilities obtained from the one year transition matrix and from historical data (source : Moody's).

As we can see, the transition matrices formalism overestimates the ten years default probabilities. However, in what follows, we are going to remain in the framework of the transition matrix formalism as explained above.

4. A CONTINUOUS TIME MODEL

4.1 A rating driven spread dynamics

For the sake of clarity, we assume that the term structure of interest rates is flat and equal to r over time. This assumption can be relaxed by specifying for instance a Vasicek like dynamics for the instantaneous rate, or a more complex model for the whole rate curve (HJM model for instance). Such a change would not change the generality of our purpose. The choice of the interest rate model is out of the scope of this paper which focuses only on the credit part.

The market is assumed to be made of two kinds of assets : a risk free asset with a constant rate of return r, and a risky zero-coupon bond with maturity T issued by a firm with rating R_t at time t. The price fluctuations of this bond are driven by two sources. First, there are market fluctuations : because risk free rate is constant, they are interpreted as spread fluctuations. Second, credit events, such as a default, can induce price variations of the risky bond.

We propose here a one factor model in order to take into account the market risk. This factor is the spread corresponding to maturity T. Our model is similar to Black's model ([3]) for interest rates since we do not build an arbitrage model for the whole term structure of the spreads. There are two main motivations for doing this : first, the model we obtain is much more simple to tackle (this is one of the reasons for the success of Black's model) ; second most of the firms have a few issues which are not sufficient to build a realistic model for the spread curve. The spread process $(X_t)_{t\geq 0}$ satisfies the following Stochastic Differential Equation (SDE) :

$$dX_t = a(R_t)[b(R_t) - X_t]dt + s(R_t)dW_t \tag{5}$$

where $(W_t)_{t\geq 0}$ is a standard brownian motion and $(R_t)_{t\geq 0}$ is the rating process of the issuer. We assume that there are D levels of rating ; the dynamics given in equation (5) is valid as long as the issuer has not defaulted $(R_t < D)$

and the coefficients of the dynamics are rating dependent. The price *B(t,T)* at time *t* of the risky zero-coupon bond with maturity *T* writes :

$$B(t,T) = e^{-(r+X_t)(T-t)} \tag{6}$$

Itô's lemma leads to the dynamics of the risky bond, and there is a unique probability change that makes the discounted price process of the bond a martingale (see appendix). Before the time of default, the spread dynamics is a mean reverting stochastic dynamics with parameters depending only on the rating level of the debt and is solution of equation (5). After the default, the rating process is assumed to remain constant because, as we can see in figure 4, the probability of coming back to a non defaulted situation is equal to zero. We assume that after the default, the risky bond turns into a riskless bond with the same maturity *T* but a decreased nominal to $\omega < 1$ (which is the recovery rate of the risky debt). Let us call $B_0(t,T)$ the price of the riskless zero-coupon bond with maturity *T*. After the default, we deduce a theoretical value for the spread *s(t)* at time *t* :

$$B(t,T) = B_0(t,T).e^{-s(t)(T-t)} = \omega.B_0(t,T) \Rightarrow s(t) = -\frac{1}{T-t}\ln\omega \tag{7}$$

We assume that the observed spread after the default is no longer stochastic and reaches its equilibrium value *s(t)* immediately after the default.

4.2 A dynamics for the rating

The specificity of this model is that it takes into account the possibility of rating transitions. It is easy to make a model for the rating process since we assume that it is a pure jump process with a finite number of possible values, and the intensities of the jumps are the coefficients of the generating matrix. The rating process $(R_t)_{t\geq 0}$ is solution of the SDE $(R_t \in \{1,..., D\})$:

$$dR_t = \sum_{i=1}^{D} (i - R_t) dN_t^i \tag{8}$$

with initial value $R_0 \in \{1,..., D\}$. The processes $(N_t^i)_{i=1..D, t\geq 0}$ are independent Poisson point processes with intensities equal to $\lambda(i, R_t) = a_{R_t,i}$ which are the coefficients of the generating matrix that describes the rating transition probabilities of the firm. Each Poisson point process $(N_t^i)_{t\geq 0}$ represents the transition between the rating level R_t (the ongoing rating level at time *t*) to the rating level *i*. Between time *t* and time *t+dt*, the probability of this rating transition is equal to the coefficient (R_t, i) of the generating matrix times the time interval *dt*. The amplitude of the rating jump when the *i*-th Poisson's process $(N_t^i)_{t\geq 0}$ jumps is $(i - R_t)$.

This model is interesting from an empirical point of view. Indeed, we deal with a jump model, but there are no estimations of the amplitudes of the jumps since they are integer numbers. Moreover, we have an estimation of the intensities of the Poisson processes thanks to the rating agencies' matrices, but these estimations do not integrate specific risk. That is why we assume that the intensities are uncertain parameters of the model, which means, from a mathematical point of view, that the equivalent martingale measure is no longer unique since the intensity of the Poisson process is not unique. This point is developed further in section 5.

4.3 Arbitrage range of option prices

In this subsection, we address the main goal of this article. We aim at computing the arbitrage range of prices of a contingent claim written on the spread. To this end, we compute the super-replication (resp. under-replication) price of options on the spread. The option is written on the spread itself or equivalently on the risky bond. Let us consider the European option with maturity *H<T* and pay-off *g(B(H,T))*. For instance the most common option on spread is the option to buy the risky bond at time *H* at a given spread ; this option is exactly a call option on the risky bond. We call π_t the proportion of a hedging portfolio invested into the risky bond at time *t*, and we consider any admissible strategies $(\pi_t)_{t\geq 0}$; we do not describe them in detail (see for instance [5]) but we just say that they are

self-financing and that they satisfy some integrability conditions. An admissible hedging portfolio is then obtained from an admissible hedging strategy. The problem of super-replication is to find the cheapest portfolio over-hedging the contingent claim. At time *t*, the value of this portfolio is :

$$\Pi_t = \inf\left[x : \exists\pi \text{ admissible} \mid V_H^{x,t,\pi} - g(B(H,T)) \geq 0 \; a.s.\right] \tag{9}$$

where $V_H^{x,t,\pi}$ is the value at time *H* of the hedging portfolio knowing that it was *x* at time *t* and we have followed strategy π. In its dual form (see Kramkov [17]), this optimization problem writes as the supremum over all equivalent martingale measures of the expected pay-off of the claim and the value of the portfolio is a function of *t*, *B*, and *R* :

$$V(t,B,R) = \sup_Q E_t^Q\left[e^{-r(H-t)} g(B(H,T)) \middle| B(t,T) = B, R_t = R\right] \tag{10}$$

Here, the supremum is taken over all the equivalent martingale measures ; in the next section, we are going to parametrize this set of measures and we shall explain which measures should be kept in the analysis. In particular, we will argue that, from a financial point of view, all equivalent martingale measures are not relevant.

5. THE RISK NEUTRAL DEFAULT PROBABILITIES

5.1 The set of uncertain transition intensities

The market is incomplete because the rating is not a negotiable asset. As shown in the appendix, the set of martingale measures can be parametrized by *D* positive real numbers $\left(p^i\right)_{i=1..D} \in S$, where the control set *S* is a subset of $\Re_*^{+D}$. These numbers are risk premiums associated to the *D* possible changes of rating. If we consider all the equivalent martingale measures, the set *S* is isomorphic to $\Re_*^{+D}$. The function *V(t,B,R)* is solution of the following non linear equation (see appendix and [16]) :

$$V_t + \frac{1}{2}s^2(R)(T-t)^2 B^2 V_{BB} + rBV_B + \sum_{i=1}^{D} \lambda(i,R) \sup_{p^i \in S}\left[p^i \Delta_{i,R} V\right] = rV \tag{11}$$

where $\Delta_{i,R}V = V(t,B,i) - V(t,B,R)$. Changing the probability measure is equivalent to changing the intensities of each Poisson process by a factor p^i. This kind of transformation on the generating matrix leads to another generating matrix for any positive value of the parameters $\left(p^i\right)_{i=1..D}$. A one year risk neutral transition matrix can then be computed. Let us illustrate this by choosing one particular risk neutral measure characterized by a set of parameters $\left(p^i\right)_{i=1..D}$. We construct the diagonal matrix $M = diag\left(p^1,..,p^D\right)$. The matrix *M.A* is still a generating matrix, and the risk neutral transition matrix between time *t* and time *t+s* is :

$$P_{RN}(t,t+s) = e^{sMA} \tag{12}$$

From a theoretical point of view, the $\left(p^i\right)_{i=1..D}$ are arbitrary strictly positive real numbers. Nevertheless, the question we address here, is to know whether they are all relevant from a financial point of view. Keeping all the equivalent martingale measures leads for instance to consider the case where the firm will default in the next minute with probability arbitrarily close to 1 (this corresponds to the limit $p^D \to \infty$). This scenario has a very important impact on the pricing procedure and affects deeply the numerical results : it is in particular responsible for the large ranges of arbitrage prices in credit models. In practice, it is not reasonable to hedge the risk that a AAA rated firm defaults in the next minute, more especially as the one year historical probability of default of a AAA rated firm is very close to zero : people are excessively prudent when they compute arbitrage prices. We argue that such a risk does not need to be hedged away in a realistic model of credit derivatives pricing (mathematically speaking, this means that the parameter p^D must not be sent to infinity).

Moreover, we expect that the risk neutral intensities are around the value obtained from the S&P matrix, the difference being equal to the specific part of credit risk. This is why we propose to bound upward the rating transition intensities instead of considering any positive real values. More precisely, our methodology is inspired from Avellaneda's model ([2], see also Wilmott in [23]) of uncertain volatility : we assume that the risk neutral rating transition intensities are not uniquely defined, but they are uncertain parameters because of the specific risk. The set of uncertain parameters is interpreted as a confidence interval on the transition intensities. This is equivalent to assume that the risk premiums are themselves uncertain parameters, belonging to the interval :

$$p^i \in \left]0, \overline{p}^i\right] \quad \forall i, \overline{p}^i \in \Re_*^+ \tag{13}$$

where the $\left(\overline{p}^i\right)_{i=1..D}$ are entries of our model. The partial differential equation (11) thus writes, for $R<D$:

$$V_t + \frac{1}{2} s^2(R)(T-t)^2 B^2 V_{BB} + rBV_B + \sum_{i=1}^{D} \lambda(i,R)\overline{p}^i (\Delta_{i,R} V)^+ = rV \tag{14}$$

This equation actually splits into *D-1* coupled equations because we have to solve the partial differential equations simultaneously for each value of the rating. In the next section, we will show this explicitly on a three rating levels model. Such non linear PDE has been first introduced in finance by Hoggard *et al.* ([13]).

5.2 Intuitive meaning of the measure bounds

The parameters $\left(\overline{p}^i\right)_{i=1..D}$ constrain the set of uncertain probabilities and then characterize the confidence interval ; these parameters have a very intuitive meaning in the special case $\overline{p}^i = \overline{p}$ (for all $i \in \{1..D\}$). We remind that the number $\overline{p}$ is a risk premium parameter for the intensities $\lambda(i,R)$. Let us write the one-year risk neutral transition probabilities :

$$P_{RN(k)}(t, t+1\,yr) = e^{kA} = P_{S\&P}(t, t+1\,yr)^k = P_{S\&P}(t, t+k\,yr), \quad k \in \left]0, \overline{p}\right] \tag{15}$$

The one year risk neutral transition matrices are simply the *k*-years S&P transition matrices with $k \in \left]0, \overline{p}\right]$. The standard and Poor's matrix is a kind of benchmark for the rating transition probabilities ; in our model we keep any generating matrix equal to a *k*-years S&P's generating matrix. The parameter $\overline{p}$ tells how prudent we are when we hedge credit risk in the pricing procedure relative to using S&P's generating matrices since it represents the maximum transition risk premium that we consider for the risky bond relative to the S&P's transition intensities. This remarkable property gives the model an intuitive understanding and a concrete criterion in order to choose the bounds on the risk neutral intensities we want to keep. For instance, choosing $\overline{p} = 500$ means that the uncertain default probabilities are between the instantaneous default probabilities and the *500*-years default probabilities : from the viewpoint of credit risk, we forget events that occur less than once in 500 years.

As it is implicitly written in eq. (14), the optimal parameters that give the upper bound are either 0 or $\overline{p}$, depending on the sign of $\Delta_{i,R}V$. When $\Delta_{i,R}V > 0$, the optimal parameter is $\overline{p}$, and it is equal to 0 in the opposite case. Moreover, the optimal parameter varies on the grid in the case when the pay-off is convex at some points and concave at others: in this case, the price of the option depends on the rating, and the sign of $\Delta_{i,R}V$ is likely to change. This is similar to what happens in Avellaneda's uncertain volatility model for spread options for instance which selects the optimal path of optimal volatility according to the sign of the gamma.

The analysis of optimal probabilities is much easier in the case of convex, concave or linear pay-offs. This later case correspond to the pricing of risky bonds and has been extensively studied by Collin-Dufresne and Hugonnier in [7]. They show that the upper price (resp. under price) corresponds to an optimal parameter equal to 0 (resp. infinity, corresponding to a default probability equal to 1). The interpretation for a linear pay-off is that, in the best case scenario, the probability of downgrade is 0, and the probability of upgrade is maximal. On the reverse case, the worst case scenario corresponds to a probability of downgrade (and thus of default) equal to 100%. In the framework we have chosen here, the upper parameter is capped to $\overline{p}$, but the same analysis remains available.

6. THREE LEVELS MODEL

In this section, we limit ourselves to a simple example of a three rating levels model. These levels are *R1*, *R2* and *D* for default. For instance *R1* stands for the investment grade rating level and *R2* stands for the speculative grade. The historical generating matrix for the transition probabilities is the one given Table 5.

Initial rating	Generating matrix (%)		
	R1	R2	D
R1	-0,3	0,2	0,1
R2	0,3	-2,33	2,02
D	0	0	0

Initial rating	Transition matrix (%)		
	R1	R2	D
R1	99,7	0,2	0,1
R2	0,3	97,7	2
D	0	0	100

Table 5 : Generating matrix and one year transition matrix in the three-level model.

In order to write down the full system of partial differential equations we need the value of the derivative after the default. This goal is easily achieved since the price of the risky bond after the default is at time t :

$$B(t,T) = \omega e^{-r(T-t)} \tag{16}$$

After the default, the value of the derivative at time t is deterministic and is the discounted final payoff :

$$V^D = V(t,B,D) = e^{-r(T-t)} g\left(\omega e^{-r(T-t)}\right) \tag{17}$$

From equation (14), the super-replication prices of the credit option for rating levels *R1* and *R2* are solution of the following system of coupled PDE :

$$\begin{cases} V_t^{R1} + \frac{1}{2} s^2 (R1)(T-t)^2 B^2 V_{BB}^{R1} + rBV_B^{R1} + \overline{p}\lambda(R2,R1)\left(V^{R2} - V^{R1}\right)^+ + \overline{p}\lambda(D,R1)\left(V^D - v^{R1}\right)^+ = rV^{R1} \\ V_t^{R2} + \frac{1}{2} s^2 (R2)(T-t)^2 B^2 V_{BB}^{R2} + rBV_B^{R2} + \overline{p}\lambda(R1,R2)\left(V^{R1} - V^{R2}\right)^+ + \overline{p}\lambda(D,R1)\left(V^D - v^{R2}\right)^+ = rV^{R2} \end{cases} \tag{18}$$

where $(A)^+ = \max(A,0)$. This system of PDE is known to have a unique solution since it is a cooperative system ([6]). It also possesses important properties of numerical convergence. The lower bound of arbitrage prices is the solution of similar equations :

$$\begin{cases} V_t^{R1} + \frac{1}{2} s^2 (R1)(T-t)^2 B^2 V_{BB}^{R1} + rBV_B^{R1} + \overline{p}\lambda(R2,R1)\left(V^{R2} - v^{R1}\right)^- + \overline{p}\lambda(D,R1)\left(V^D - v^{R1}\right)^- = rV^{R1} \\ V_t^{R2} + \frac{1}{2} s^2 (R2)(T-t)^2 B^2 V_{BB}^{R2} + rBV_B^{R2} + \overline{p}\lambda(R1,R2)\left(V^{R1} - v^{R1}\right)^- + \overline{p}\lambda(D,R1)\left(V^D - v^{R2}\right)^- = rV^{R2} \end{cases} \tag{19}$$

where $(A)^- = \min(A,0)$. We have computed the numerical solution for the call option on the spread : this option gives the right to buy the risky bond at a given spread (the strike of the option) at the maturity. In figure 1, we give the curves of the range of prices for this option with strike 0 basis points and maturity 3 months ; the rating of the issuer at time 0 is *R1*. We have chosen the confidence interval for the uncertain parameters : $\overline{p} = 500$.

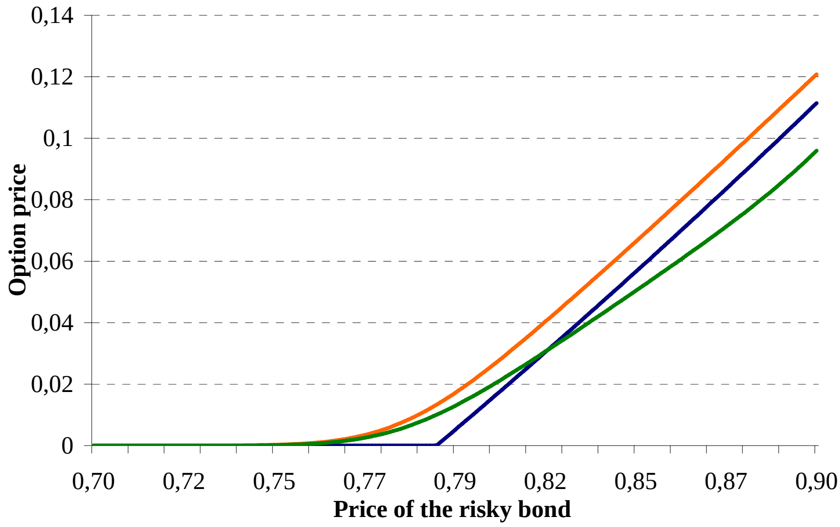

Figure 1 : Arbitrage range of prices for a call option on the spread of a risky bond with strike 0 bp and maturity 3 months.

As we can see, the arbitrage range of prices is quite narrow in this model in spite of a very large value of the parameter $\overline{p}$. The curves cross the payoff function contrary to Black-Scholes' model : the possibility of a credit event makes the in-the-money prices of the European call options lower than the pay-off itself.

Let us now turn to the call spread option on the spread of the risky bond. This option is actually a call spread option on the risky bond itself. This option gives the right to buy the risky bond at a spread equal to $\max(strike1, X(H) - strike2)$. In the language of risky bonds, this is exactly a call spread on the risky bond. For numerical applications, we consider the call spread option with strike 1 equal to 0, strike 2 equal to 25 bp and maturity 3 months ; the confidence interval of the transition intensities is still defined by $\overline{p} = 500$.

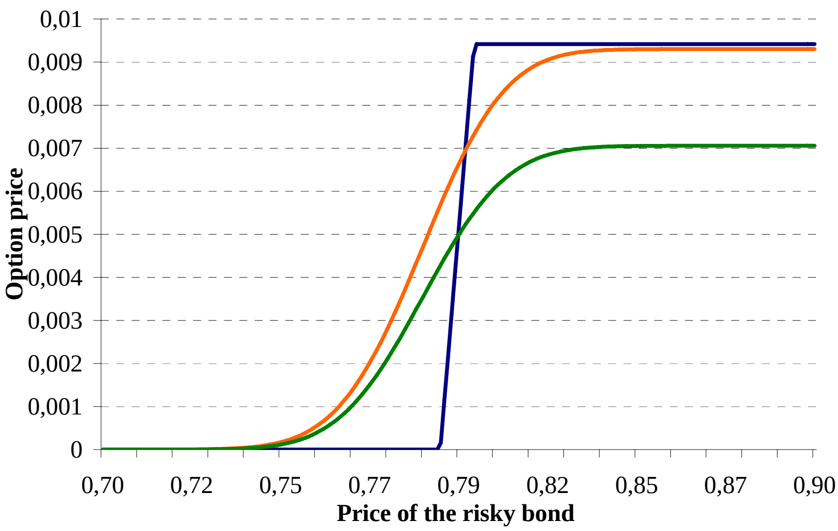

Figure 2 : Arbitrage range of prices for a call spread option on the spread with strikes 0 basis points and 25 basis points respectively, and maturity 3 months.

7. CONCLUSION

We have described a simple model for the evolution of credit spreads based on credit ratings transitions. This model is appropriate for pricing European options on spread. The main interest of this article is not in the spread model itself, but rather lies in the choice of the risk-neutral probabilities which are mapped on the set of the uncertain transition intensities. We assume that the default probabilities do not explain the whole spread, and on the reverse, the spread is not enough to choose the risk neutral transition probabilities.

Our model is an uncertain probability model and leads to a range of arbitrage prices for credit derivatives since we give a confidence interval for the choice of the uncertain intensities. We have then computed numerically this range of prices and we have shown that it was quite narrow for short term options.

This uncertain probability model is similar to Avellaneda's uncertain volatility model ([2]) but leads to a system of non linear partial differential equations for the extreme bounds of the arbitrage prices. However, the interpretation of the probability interval is quite similar : we interpret it as a confidence interval for the specific rating transition probabilities.

APPENDIX : DERIVATION OF THE NON LINEAR PDE

For the sake of clarity, we only provide the reader with a sketch of a derivation of the non linear partial differential equation (14). For a mathematically precise proof of these results, we refer to [5] and [8]. The super-replication price of the contingent claim with payoff *g(B(H,T))* and maturity *H*<*T*, is :

$$V(t,B,R) = \sup_Q E_t^Q\left[e^{-r(H-t)} g(B(H,T)) \middle| B(t,T) = B, R_t = R\right] \tag{A1}$$

Changing the probability measure leads to introduce a risk premium for the brownian part (*i.e.* a process $(\theta_t)_{t\geq 0}$ and a risk premium for the jump part (one risk premium process $\left(p_t^i\right)_{t\geq 0, i=1..D}$ per jump process). The change of probability measure writes :

$$\begin{aligned} d\tilde{W}_t &= dW_t + \theta_t dt \\ d\tilde{M}_t^i &= dM_t^i + \left(1 - p_t^i\right)\lambda(i, R_t)dt \quad (i = 1..D) \end{aligned} \tag{A2}$$

Under an equivalent measure, the discounted risky bond price is a martingale. The risk premium $(\theta_t)_{t\geq 0}$ linked to the brownian motion is then:

$$\theta_t s(R_t)(T-t) = X_t - a(R_t)(T-t)[b(R_t) - X_t] + \frac{1}{2} s(R_t)^2 (T-t)^2 \tag{A3}$$

The risk premiums $\left(p_t^i\right)$ associated to the jump processes are chosen to be markovian in order to go on with a markovian model. In general, we make this assumption in order to use Bellman's principle. We assume that the nature of the results remains unchanged whereas the calculations are simplified (see for instance [8]). The processes of the price of the risky bond (before default) and rating write :

$$\begin{cases} \dfrac{dB}{B} = rdt - s(R_t)d\tilde{W}_t \\ dR_t = \displaystyle\sum_{i=1}^{D} (i - R_t)\lambda(i, R_t)p_t^i dt + \sum_{i=1}^{D} (i - R_t)d\tilde{M}_t^i \end{cases} \tag{A4}$$

The processes $\left(p_t^i\right)$ are free parameters of our model, and they parameterize the set of equivalent martingale measures ; we denote the equivalent martingale measures as $Q^{p^{\&}..p^{D}}$. Thanks to the markovian nature of all the processes under the equivalent martingale measures, we can apply Bellman's principle between time *t* and *t+h*, which leads to the equation :

$$\sup_{\left(p_s^i\right)_{i=1..D,t\le s\le t+h}} \frac{1}{h} E_{t,X,R}^{Q^{p^1..p^D}}\left[\int_t^{t+h} DVds\right]=0 \tag{A5}$$

where *DV(t,B,R)* is the Dynkin operator applied to the function *V(t,B,R)*. In the limit $h\to 0$, the integral converges to the value of the integrand at time *t*, and the supremum over all the processes $\left(p_t^i\right)$ is transformed into a supremum over real parameters (value of the processes at time *t*). Lets us call *S* the Euclidean sub-space of $\Re^D$ in which the *D*-varied process $\left(p_t^1,..,p_t^D\right)_{t\ge 0}$ takes its values. The Dynkin of the function *V(t,B,R)* is obtained thanks to Itô's lemma extended to the case of jumping processes. Applying Itô's lemma to the function *V(t,B,R)* leads :

$$dV=V_t+V_B dB+\frac{1}{2}V_{XX}\,d\langle B,B\rangle_t+\sum_{i=1}^{D}\Delta_i V\left[\lambda(i,R_t)p^i dt+d\tilde{M}_t^i\right] \tag{A6}$$

where $\Delta_i V=V(t,B,r,i)-V(t,B,r,R)$. Inserting this result into eq. (A5), we obtain the partial differential equation satisfied by the super-replication price *V(t,B,R)* :

$$V_t+\sup_{p^i\in S}\left[\frac{1}{2}s(R)^2(T-t)^2B^2V_{BB}+rBV_B+\sum_{i=1}^{D}\lambda(i,R)p^i\Delta_i V\right]=rV \tag{A7}$$

This is equation (11). At this stage, the control set *S* is not yet specified. In the article, we did not take the whole set of equivalent martingale measures, and we have bounded the control set *S* to the rectangle $\left[0,\bar{p}^1\right]\times\cdots\times\left[0,\bar{p}^D\right]$.

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