Super-replication problem in a jumping financial market

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Problème de surcouverture dans un marché à sauts

Résumé

Les marchés incomplets constituent un problème important de la finance théorique car on ne peut couvrir tous les actifs contingents à partir des actifs du marché ; cependant, il est toujours possible de les sur-couvrir. Cet article s'intéresse au problème de sur-couverture des options européennes lorsque les prix des actifs risqués et/ou leur volatilité peuvent effectuer des sauts à des dates aléatoires. Nous explorons une partie de l'ensemble des mesures martingales équivalentes qui correspondent à des changements de probabilités markoviens. Sur ce sous-ensemble des mesures martingales équivalentes, nous montrons que le prix maximum pour les actifs contigents est solution (au sens de la viscosité) d'une équation de Hamilton-Jacobi-Bellman non linéaire. Nous résolvons cette équation dans de nombreux cas et exhibons des bornes de prix non triviales lorsqu'une solution analytique explicite ne peut être trouvée.

Super replication problem with a jumping financial market

Abstract

Incomplete markets are known to worry theoricians because not every contingent claim is replicable ; however, if one can afford it, it is possible to super-replicate. This paper deals with the super-replication problem of any European contingent claim when stock prices and/or volatilities may jump at random dates. We explore a subset of all the equivalent martingale measures that correspond to markovian changes of probability. On this subset, we show that the maximum price for a contingent claim is the viscosity solution of some non linear Hamilton-Jacobi-Bellman equation that can be solved in most cases and we obtain non trivial bounds of the super-replication price in the other cases.

Introduction

In the framework of complete financial markets, it is well known that any contingent claim can be replicated by an appropriate investment strategy in the assets of the market (Harrison and Kreps, 1979 ; Harrison and Pliska, 1981). This is the main characteristic of Black-Scholes' (BS) model ; then, the principle of absence of arbitrage opportunity guarantees a unique price for the contingent claim because of the existence of a unique risk-neutral probability.

In the context of incomplete markets modeled by continuous time diffusions, there are more sources of randomness than negotiable assets, so that all random sources cannot be hedged away by a dynamic trading (Follmer and Schweizer, 1990). This raises very conceptual questions and among them is the pricing problem : the price of a contingent claim is no longer uniquely defined when the market is incomplete because incompleteness, from a mathematical viewpoint, generates an infinity of equivalent martingale measures (El Karoui and Quenez, 1995). For the time being, three main approaches of this question exist. The first one is to choose a particular equivalent martingale measure, for instance by minimizing the quadratic risk ; the second one is to build a utility dependent strategy ; the third one is to build a super-hedging (Broadie et al., 1996 ; Cvitanic and Karatzas, 1996 ; Jouini and Kallal, 1995) strategy whose price is the largest price obtained with all the equivalent martingale measures (see Jouini and Kallal, 1995 ; Cvitanic and Karatzas, 1996).

Stochastic volatility models (Avellaneda et al., 1995) are amongst the most popular of incomplete market models because they forget the unrealistic assumption of constant volatility in BS' model. Transaction costs have also been extensively studied (Avellaneda and Paras, 1994 ; Davis et al., 1993 ; Soner et al., 1995) because they are an obvious obstacle to completeness since they act as a viscous force on the nice Black-Scholes mechanics. A third example of incomplete markets are markets in which stock prices may jump at random times (Jeanblanc and Pontier, 1990). Merton (Merton, 1976) first introduced such a model in 1976, and this was very relevant in order to describe shocks undergone by price processes, for instance after the publication of some economical data or political decision. Jump models are thus particularly relevant for an exchange and interest rate market.

Paradoxically, jump models have been quite ignored in the financial literature, maybe because they are expected to behave similarly to other kind of incomplete market models. This last belief is partially justified if we consider the super-replication problem. We expect the price of the cheapest portofolio super-hedging a european contigent claim to depend only marginally on the type of incompleteness. However, some important differences may appear because of the specificity of each model, and jump models are important enough to be studied.

In this article, we argue that the super-replication price of a contingent claim is the supremum over all the equivalent martingale measures set of the expected cash-flow. We show that if we restrict the set of equivalent martingale measures to the markovian changes of probability, the problem becomes mathematically tractable and gives some interesting insights on the super-replication problem ; indeed, in all the known cases, this "super price" is equal to the super-replication price. In the whole article, we call "super price" the maximum price of a contingent claim over all the markovian equivalent martingale measures. We give an analytical solution to the "super price" computation for a European contingent claim in the context of jumping underlying assets and volatilities. We write this problem as an optimization problem, and use standard stochastic control theory and dynamic programming to solve it. The "super price" is expressed as a solution of a non linear integro-differential equation that must be taken in the sense of viscosity (Crandall and Lions, 1983 ; Crandall et al., 1992) because the existence of a regular solution cannot be proved.

The results we obtain are not surprising ; however, it is interesting to get a proof of these results, more especially as the Hamilton-Jacobi-Bellman (HJB) equation involved here has an unusual additional term (which is a jump term) compared to stochastic volatility and transaction costs models. Because of this integro-differential additional jump term, the equation is more complicated to solve (and it is not always possible to do it). In the general case, when the underlying asset and its volatility both jump, we obtain non trivial bounds for the super price of any European contingent claim ; we even solve the problem exactly when the stochastic jumping volatility is not bounded. When only the volatility jumps, we exactly recover the results of continuous stochastic volatility, and this is necessary to be derived.

This article is organized as follows : in section 1, we describe the market model including jumps and we define the super-replication problem as an optimization problem over all equivalent martingale measures. In section 2, we identify and parametrize all the equivalent martingale measures ; then, we restrict the set of martingale measures to the set of markovian probability changes so that we get a well-defined (but singular) control problem. Section 3 is devoted to obtaining the HJB equation satisfied by the super price, and some properties of the result function are derived ; this allows to simplify the HJB equation. Finaly, in section 4, we solve this equation when possible or, at least, we give bounds on the solution.

1 The model

In comparison with BS model, jump models admit an additional term in the stochastic differential equation (SDE) which solution is the price process of the risky asset. The new ingredient entering jump processes is the Poisson point process $(N_t)_{t\geq 0}$.

Let $W = \{W_t, \mathcal{F}_t^W; 0 \leq t \leq T\}$ be a Wiener process defined on the probability space $(\Omega^W, \mathcal{F}^W, P^W)$ where $\mathcal{F}^W$ is the P^W-augmentation of the filtration generated by W. The process $N = \{N_t, \mathcal{F}_t^N; 0 \leq t \leq T\}$ is a Poisson process of deterministic and bounded intensity $\lambda(t)$ defined on the space $(\Omega^N, \mathcal{F}^N, P^N)$ where $\mathcal{F}^N$ is the P^N-augmentation of the filtration generated by N. We now consider the space $(\Omega, \mathcal{F}, P) = (\Omega^W \otimes \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, P^W \times P^N)$ in which W and N are independent by construction.

Let us consider a market model in which there are only a riskless asset with constant price $S_t^0 = 1$ for all $t \geq 0$ (the assumption of zero interest rate can here be replaced by discounting ; this is not prejudiciable to the generality of the purpose) and one risky asset S^1. We assume that the dynamics is driven by a two-factors model. We also assume that the dynamics of the risky asset depends on another asset S^2 that is not negotiable. For instance the asset S^2 can be the unemployement rate, and it is realistic to build a 2-factors model in which the dynamics of the risky asset S^1 also depends on the unemployment rate S^2. We call the non negotiable asset S^2 the fictitious asset. In this article, we then assume that the price process of the asset S^1 is solution of the following SDE :

$$\frac{dS_t^1}{S_{t^-}^1} = \mu_1(t, S_t^1, S_t^2)\, dt + \sigma_1(t, S_t^1, S_t^2)\, dW_t + \phi_1(t)\, dN_t,$$

where the process (S_t^2) is solution of the SDE :

$$\frac{dS_t^2}{S_{t^-}^2} = \mu_2(t, S_t^1, S_t^2)\, dt + \sigma_2(t, S_t^1, S_t^2)\, dW_t + \phi_2(t)\, dN_t.$$

The function μ_1 is a deterministic function of S^1 and S^2 ; ϕ_1 and ϕ_2 are choosen to be deterministic functions, and the positivity of the price of the negociable asset S^1 requires $\phi_1(t) > -1$ for all $t \geq 0$. We also take $\phi_2(t) > -1$ which implies that (S_t^2) is also a price process, the price process of a fictitious asset S^2 ; this does not change the generality of the purpose. The volatilities σ_1 and σ_2 are deterministic functions of the prices S_1 and S_2.

These SDE always involve the factor $S^i_{t^-}$ $(i = 1, 2)$ instead of S^i_t so that the processes (S^i_t) are *càdlàg* processes. In this model, the volatility σ_1 itself can also jump. When initial conditions are specified, existence and unicity of a strong solution (S^1_t, S^2_t) in the case of jump processes are guaranteed by well known theorems, provided that the coefficients of the SDE are sufficiently regular. To this end, we suppose the coefficients of the jump terms $\phi_1(t)$ and $\phi_2(t)$ to be continuous. Drifts and volatilities are taken continuous in (t, s_1, s_2) and the functions $s_i\, \mu_i(t, s_1, s_2)$, $s_i\, \sigma_i(t, s_1, s_2)$ and $\sigma_i^{-1}(t, s_1, s_2)$ $(i = 1, 2)$ are lipschitz relative to s_1 and s_2 uniformly in t. Such conditions are sufficient for existence and uniqueness (Protter, 1990, p.197), and they will allow later to apply the principle of dynamic programming.

We assume furthermore the coefficients of the preceeding SDE to satisfy the following relations :

$$|\sigma_1\phi_2 - \sigma_2\phi_1| \geq \delta > 0, \quad a.s., \quad \delta > 0. \tag{1}$$

$$\frac{\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_1\phi_2 - \sigma_2\phi_1} > 0 \;\; a.s. \tag{2}$$

The first relation enforces the fictitious asset S_2 not to be redundant with S_1 ; in other words, the random dependence of both assets are not proportional. The second relation enforces the intensity of the Poisson process to be positive under any equivalent martingale measure ; it can be shown that if the asset S_2 were negotiable (Jeanblanc and Pontier, 1990), the market composed of the riskless asset S_0 and these two risky assets S_1 and S_2 would be complete, given the two preceeding assumptions.

A trading strategy $(\pi_t)_{t\geq 0}$ is a predictable $\mathcal{F}$-adapted left-continuous process on $[0, T]$ satisfying an integrability condition $\mathbf{E}\int_0^T \pi_u^2 \sigma_1^2(u, S_u^1, S_u^2) S_1^2(u) du < \infty$. Here, π_t is the number of risky assets owned at time t. The value of the portfolio at time t is :

$$V_t = \pi_t S_t^1 + (V_t - \pi_t S_t^1) S_t^0 = \pi_t S_t^1 + (V_t - \pi_t S_t^1). \tag{3}$$

The self-financement condition writes :

$$dV_t = \pi_t dS_t^1. \tag{4}$$

In what follows, we will consider a European contingent claim with pay-off $g(S_T^1)$ at maturity T, and we will study the problem of super-replication of this claim.

We note $V_u^{x,t,\pi}$ the value at time $u \geq t$ of the portfolio that had value x at time t and given the strategy $(\pi_s)_{t\leq s\leq u}$. An admissible portfolio is obtained from an admissible strategy that belongs to the set $\mathcal{A}(x,t) = \left\{\pi : V_u^{x,t,\pi} \geq -C,\ t \leq u \leq T \text{ a.s.}\right\}$ (C is a positive constant). The problem of super-replication is to calculate the cheapest portfolio over-hedging the contingent claim. At time t, this portfolio value is :

$$\Pi_t = \inf\left\{x\ :\ \exists \pi \in \mathcal{A}(x,t),\ V_T^{x,t,\pi} - g(S_T^1) \geq 0 \text{ a.s.}\right\}$$

In its dual form (see Kramkov 1996), this optimization problem writes as the supremum over all equivalent martingale measures of the expected value of the claim :

$$\Pi_t = V(t, s_1, s_2) = \sup_Q \mathbf{E}^Q\left[g(S_T^1) | (S_t^1, S_t^2) = (s_1, s_2)\right],$$

where s_1 and s_2 are the values of S_1 and S_2 at time t. Here, the supremum is taken over all equivalent martingale measures ; they will be clearly parametrized in the next section. This formulation includes the Markov property of the price processes.

Bellamy and Jeanblanc (1998), Eberlein and Jacod (1997) have given the range of option prices in the case when the pay-off profile $g(x)$ is convex, positive and $g(x)/x$

is bounded. Here we solve the super price problem for any positive pay-off function. Sometimes, the solution will require the concave envelope of g ; we will assume, when this occurs, that $g(x)/x$ is also bounded.

2 Formulation of the control problem

In order to get a tractable control problem, we need to identify and parametrize the equivalent martingale measures that preserve the Markov property of the prices, and Girsanov's theorem is adapted to the case of jump processes in appendix A. All the equivalent probabilities Q are such that the only negotiable asset price process (S_t^1) is a Q-martingale.

The Radon-Nikodym derivative leading to an equivalent probability measure Q is :

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = L_t \,,$$

with (L_t) choosen markovian :

$$dL_t = L_{t^-}\left[-\theta(t, S_t^1, S_t^2, \mu_2, \sigma_2, \phi_2)dW_t + (p(t, S_t^1, S_t^2, \mu_2, \sigma_2, \phi_2) - 1)(dN_t - \lambda(t)dt)\right].$$

So an equivalent probability measure is well defined when we are given the processes θ and p ; they are the risk premiums for the brownian part and the jump part respectively. The requirement of the martingale property for the negotiable risky asset, implies a relationship between the processes θ and p, so that the equivalent martingale measures can be parametrized by the process p only :

$$\mu_1(t, S_t^1, S_t^2) - \sigma_1\theta(t, S_t^1, S_t^2) + \phi_1\lambda\left[p(t, S_t^1, S_t^2) - 1\right] = 0.$$

The process p is thus a control variable of the problem, and so are μ_2, σ_2 and ϕ_2. These four parameters are not redundant because the process p can be choosen independently from the three others. In what follows, we will denote the equivalent martingale measures by $Q^{\mu_2,\sigma_2,\phi_2,p}$ and $K = \mathbb{R} \times \mathbb{R}^+ \times]-1;+\infty[\times\mathbb{R}^+$ is the subset of Euclidean space in which μ_2, σ_2, ϕ_2 and p take their values respectively. Under the probability measure $Q^{\mu_2,\sigma_2,\phi_2,p}$, the new Wiener process is written $\tilde{W}_t$ and N_t is a Poisson process of intensity λp (Jeanblanc and Pontier, 1990). Here we make the following assumption :

Assumption [A] : The process p is a deterministic function, i.e. $p_t = p(t)$.

The local martingale processes that model randomness under the historical probability write as follows under the probability $Q^{\mu_2,\sigma_2,\phi_2,p}$:

$$d\tilde{W}_t = dW_t + \theta_t dt, \tag{5}$$

$$d\tilde{M}_t = dM_t + (1-p_t)\lambda_t dt = dN_t - \lambda_t p_t dt. \tag{6}$$

Under this change of probability, the price processes involved in this problem satisfy the following SDEs :

$$\frac{dS_t^1}{S_{t^-}^1} = \sigma_1 d\tilde{W}_t + \phi_1 d\tilde{M}_t, \tag{7}$$

$$\frac{dS_t^2}{S_{t^-}^2} = (\mu_2 - \sigma_2\theta + \phi_2\lambda p)dt + \sigma_2\, d\tilde{W}_t + \phi_2\, d\tilde{M}_t \tag{8}$$

Under the historical probability, the 2-dimensional process $\left(S_t^1, S_t^2\right)$ is markovian (see Protter, 1991). If assumption [A] holds, the processes $\tilde{W}$ and $\tilde{M}$ are independant (see Protter, 1991, theorem 32 P.238), and, under a change of probability, the process $\left(S_t^1, S_t^2\right)$ remains markovian. The control problem for super-replication is thus thoroughly defined :

$$\Pi_t = V(t, s_1, s_2) = \sup_{Q^{\mu_2,\sigma_2,\phi_2,p}} \mathbf{E}^{\mu_2,\sigma_2,\phi_2,p}\left[g(S_T^1)|(S_t^1, S_t^2) = (s_1, s_2)\right]. \tag{9}$$

Under assumption [A], we solve this optimisation problem explicitly thanks to Bellman's principle. The optimum reached for markovian changes of probabilities has been shown to be the solution of the super-replication problem in the case of the transaction costs problem (see Cvitanic and al., 1998) and in the case of stochastic volatility models under portfolio constraints (see Cvitanic and al., 1999). That is why we solve the optimisation problem over all the markovian changes of probabilities only. Our aim is to express the Bellman equation associated with this problem and to extract some properties of the function $V(t, s_1, s_2)$; this will turn out to be sufficient to determine V explicitly in most cases. In fact, the Bellman equation is a non linear differential equation and there is no reason why a regular solution, in the classical sense, would exist. This is why we will consider this equation in the viscosity sense.

3 The HJB equation and its consequences

The theory of viscosity solutions (Crandall and Lions, 1983 ; Crandall et al., 1992 ; Fleming and Soner, 1993) has been designed in order to get some results from non linear partial differential equantions. Let us consider the Lower Semi-Continuous (LSC) envelope V_* of V, *ie* the largest lower semi-continuous function which is smaller than V, given by :

$$V_*(t, x, y) = \liminf_{(t_n, x_n, y_n) \to (t, x, y)} V(t_n, x_n, y_n). \tag{10}$$

We then have the following property of function $V_*(t, x, y)$:

Theorem 1 *The function $V_*(t, s_1, s_2)$ is a super-solution LSC on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$ of the HJB equation :*

$$\inf_K \left[-\mathcal{L}v - \mathcal{G}v - \mathcal{J}v\right] = 0,$$

with a boundary condition :

$$V_*(T^-, s_1, s_2) = g_*(s_1),$$

where :

$$\mathcal{L}v = v_t + \frac{1}{2}\sigma_1^2\, s_1^2 v_{s_1 s_1} \tag{11}$$

$$\mathcal{G}v = \left[\mu_2 - \frac{\sigma_2}{\sigma_1}(\mu_1 + \lambda p \phi_1) + \lambda p \phi_2\right] s_2 v_{s_2} + \frac{1}{2}\sigma_2^2 s_2^2 v_{s_2 s_2} + \sigma_1 \sigma_2 s_1 s_2 v_{s_1 s_2} \tag{12}$$

$$\mathcal{J}v = \lambda p \left[\delta v - \phi_1 s_1 v_{s_1} - \phi_2 s_2 v_{s_2}\right] \tag{13}$$

$$\delta v = v(t, s_1(1+\phi_1), s_2(1+\phi_2)) - v(t, s_1, s_2) \tag{14}$$

$g_*(\cdot)$ *is the lower semi-continuous envelope of* g*.*

Proof :

The proof of this theorem involves two preliminary results of the theory of processes.

Lemma 1 *Let* (t^n, s_1^n, s_2^n) *be a sequence that converges to* (t, s_1, s_2) *and* $\theta \geq t^n$ *for all* n *a stopping time. Then, there exists a subsequence of this one such that :*

$$\left(S^1_{t^n, s_1^n, s_2^n}(\theta), S^2_{t^n, s_1^n, s_2^n}(\theta)\right) \xrightarrow{a.s.} \left(S^1_{t, s_1, s_2}(\theta), S^2_{t, s_1, s_2}(\theta)\right)$$

where $S^i_{t,s_1,s_2}(\theta)$ *is the price process at time* θ *that was equal to* s_i *at time* t*.*

In the case when the processes are pure geometric brownian motions, this is a classical result that comes from Gronwall's lemma. If jumps can occur, this result is easy to prove in the case of constant coefficients in the SDE because exact formulas exist for these equations. The general case of stability of the solution of a SDE, when we perturb the initial conditions, is also a classical result, that can be found in Protter's book (Protter, 1990, p. 246).

This lemma is necessary in order to show the following result :

Lemma 2 (Dynamic programming principle) *We assume that all the processes are* *markovian and let θ be a stopping time such that $t \leq \theta \leq T$. Then :*

$$V_*(t,s_1,s_2) \geq \sup_{Q^{\mu_2,\sigma_2,\phi_2,p}} \mathbf{E}^{\mu_2,\sigma_2,\phi_2,p}_{t,s_1,s_2}\left[V_*(\theta), S^1(\theta), S^2(\theta))\right]$$

where the upper indices represent all the equivalent martingale measures, and the lower ones are the initial conditions.

This result is also well known in stochastic control theory (for a detailled proof, see for instance Cvitanic et al., 1997 and 1998). We are now able to perform the proof of the theorem.

Let $(t,s_1,s_2) \in [0,T] \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ and φ a function of $\mathcal{C}^2\left([0,T] \times \mathbb{R}^*_+ \times \mathbb{R}^*_+\right)$ such that :

$$0 = (V_* - \varphi)(t,s_1,s_2) = \min_{[0,T]\times\mathbb{R}^*_+\times\mathbb{R}^*_+} (V_* - \varphi)(u,x,y).$$

We replace V_* by φ in the inequality of the preceeding lemma and we get for $\theta = t + h \wedge T_n$:

$$0 \geq \sup_K \mathbf{E}^{\mu_2,\sigma_2,\phi_2,p}\left[\varphi(t+h\wedge T_n, S^1_{t+h\wedge T_n}, S^2_{t+h\wedge T_n}) - \varphi(t, S^1_t, S^2_t)\right], \tag{15}$$

where T_n is an increasing sequence of stopping times, for instance $T_n = \inf\{u \geq t,\ S^1_u \geq (n+1)S^2_t\}$.

As we can see, the expression inside the expectation is just the integral between t and $t + h \wedge T_n$ of the differential of φ. Itô's lemma can be generalized to the case when jumps can occur (see appendix) and we get this differential :

$$d\varphi = (\mathcal{L}\varphi + \mathcal{G}\varphi + \mathcal{J}\varphi)\, dt + (\cdots)d\tilde{W}_t + (\cdots)d\tilde{M}_t. \tag{16}$$

where all the terms are given in the theorem.

The expectation value of the random terms is null because of their martingale property, so we end up with :

$$\forall(\mu_2, \sigma_2, \phi_2, p) \in K, \ \ \frac{1}{h}\mathbf{E}^{\mu_2,\sigma_2,\phi_2,p}_{t,s_1,s_2}\left[\int_t^{t+h\wedge T_n} du\,(-\mathcal{L}\varphi - \mathcal{G}\varphi - \mathcal{J}\varphi)\right] \geq 0 \tag{17}$$

By taking the limit inf $h \to 0$, the integral tends to the integrand evaluated at time t. As a conclusion :

$$\inf_{\mu_2,\sigma_2,\phi_2,p}(-\mathcal{L}\varphi - \mathcal{G}\varphi - \mathcal{J}\varphi) \geq 0 \tag{18}$$

The boundary condition (at time T) is obtained as follows : we take the limits in Eq. (9) as t approaches T and we see that :

$$\liminf_{t\to T} V(t, s_1, s_2) \geq g(s_1)$$

by Fatou's lemma and by lemma 1. The boundary condition on the function $V_*(t, s_1, s_2)$ in theorem 1 comes from the definition of the lower semi-continuous envelope.

□

The main difference on regard with the case of continuous stochastic volatility or transaction costs models is the presence here of an additional jump term $\mathcal{J}v$ in the HJB equation. This term is an integro-differential term in the variables s_1 and s_2 ; thus, at first sight, the hope to solve this HJB equation seems to vanish. We shall see in the next section that this is not always the case : we will at least be able to give bounds on V_*.

Here, we only have shown the property of viscosity super-solution for V_*. This is because our optimal control problem is singular (the control set is non compact ; see Fleming and Soner 1993). This approach requires very weak regularity assumptions on V_*, and this will turn out to be sufficient for the following. The next thing to do is to explore the control set and play with all the degrees of freedom of our model ; it is then possible to show that the HJB equation can be simplified.

Lemma 3 *The function V_* is independent of s_2.*

Proof :

This proof is very similar to the proof proposed by Cvitanic et al. (1997, 1998). Let $(t, s_1, s_2) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ and φ a function of $\mathcal{C}^2\left([0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*\right)$ such that :

$$0 = (V_* - \varphi)(t, s_1, s_2) = \min_{[0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*} (V_* - \varphi).$$

For all $(\mu_2, \sigma_2, \phi_2, p) \in K$, we have :

$$-\mathcal{L}\varphi - \mathcal{G}\varphi - \mathcal{J}\varphi \geq 0, \tag{19}$$

We then send respectively $\mu_2 \to +\infty$ when φ_{s_2} is positive and $\mu_2 \to -\infty$ when φ_{s_2} is negative ; in order to satisfy the HJB inequation, we must have :

$$\varphi_{s_2} = 0. \tag{20}$$

This implies in particular that $V_*(t, s_1, s_2)$ is a lower semi-continuous viscosity supersolution of the equation $v_{s_2} = 0$, and we deduce (see Cvitanic et al., 1998, lemma 5.3) that $V(t, s_1, .)$ is a viscosity supersolution of the same equation for any fixed (t, s_1). We fix (t, s_1) and omit them in the following of the proof. We also fix y_0, y_2. We consider $y_0 < y_1 < y_2$ and a test function φ such that :

$$(V_* - \varphi)(y_1) = 0 = \min_{y_0 \leq s_2 \leq y_2} (V_* - \varphi)(s_2). \tag{21}$$

We conclude that $\varphi_{s_2}(y_1) = 0$. Since the constant function $v = V_*(y_2)$ is also a solution of the equation $v_{s_2} = 0, \; s_2 \in [y_0, y_2], \; v(y_2) = V_*(y_2)$, we get by the maximum principle (see Crandall et al., 1992, Theorems 3.3 and 8.2) :

$$V_*(s_2) \geq V_*(y_2), \;\; y_0 \leq s_2 \leq y_2. \tag{22}$$

Since y_0 and y_2 are arbitrary, V_* is non increasing. We prove the opposite inequality by defining :

$$W(s_2) = V_*(y_2 + y_0 - s_2), \;\; y_0 \leq s_2 \leq y_2. \tag{23}$$

Let $y_1 \in [y_0, y_2]$ and a $\mathcal{C}^1$ test function ψ such that :

$$(W - \psi)(y_1) = 0 = \min_{y_0 \leq s_2 \leq y_2} (W - \psi)(s_2). \tag{24}$$

Then, defining the $\mathcal{C}^1$ test function φ by :

$$\varphi(s_2) = \psi(y_2 + y_0 - s_2) y_0 \leq s_2 \leq y_2, \tag{25}$$

we see that :

$$(V_* - \varphi)(y_2 + y_0 - y_1) = 0 = \min_{y_0 \leq s_2 \leq y_2} (V_* - \varphi)(s_2) \tag{26}$$

by a change of variable. Thus, we must have $\psi_{s_2}(y_2 + y_0 - y_1) = -\varphi_{s_2}(y_1) = 0$. It follows that W is a supersolution of the equation :

$$v_{s_2} = 0, \; s_2 \in [y_0, y_2], \; v(y_2) = V_*(y_0). \tag{27}$$

The above argument gives $W(s_2) \geq W(y_2)$ or $V_*(s_2) \geq V_*(y_0)$ for $s_2 \in [y_0, y_2]$. Since both y_0, y_2 are arbitrary, we conclude that V_* is also nondecreasing and hence does not depend on s_2.

□

We have used the degree of freedom we had on μ_2. In the following lemma, we use the degree of freedom we have on the control variable p.

Lemma 4 *The HJB equation can thus be simplified : $V_*(t, s_1)$ is a solution of the equation :*

$$\min\left[-\inf_K \mathcal{L}v, -\inf_K \mathcal{J}v\right] \geq 0.$$

where the jump term is $\mathcal{J}v = \delta v - \phi_1 S_1 v_{s_1}$.

Proof :

We first forget the dependence in the variable s_2. We then choose a test function φ as usual ; the HJB equation writes for each test function :

$$\inf_K \left[-\mathcal{L}\varphi - \lambda p \left(\delta\varphi - \phi_1 S_1 \varphi_{s_1}\right)\right] = 0, \tag{28}$$

If we take the limit $p \to 0$, we get :

$$\inf_K [-\mathcal{L}\varphi] \geq 0 \tag{29}$$

Similarly, for $p \to +\infty$,

$$\inf_K [-\mathcal{J}\varphi] \geq 0 \tag{30}$$

A more compact formulation gives the proof of the lemma.

□

The jump term still resists to all these simplifications. Here, it is re-expressed as an additional constraint that the super-replication price must necessarily satisfy. If this constraint were absent, we would obtain the well-known solvable HJB equation of Cvitanic et al. (Cvitanic et al. 1997). A sufficient condition on the solution of this known equation so that the constraint is satisfied is that V_* is concave in s_1 ; this condition is however not necessary.

4 Closed form solution to the HJB equation

At first sight, the HJB equation is quite difficult to handle because of the presence of jump terms. However, it is possible to get a non trivial result for the super-replication price. We will first study the case where the volatility of the asset S_1 is deterministic, but its price can jump at random dates ; then the case of a real geometric brownian motion, but with jumping stochastic volatility, will be considered. Finaly, we will consider the general case of jumping price and volatility.

4.1 Deterministic volatility

When the volatility is deterministic, or more generally when σ_1 is only a function of t and s_1, we can show that the super price is larger than the solution of the BS type equation with volatility σ_1. In this particular case, Eq. (29) tells that V_* is a solution of :

$$\begin{aligned} & \inf_K(-\mathcal{L}v) = -\mathcal{L}v \geq 0, \\ \Leftrightarrow \quad & -v_t - \frac{1}{2}\sigma_1^2 s_1^2 v_{s_1 s_1} \geq 0, \end{aligned} \tag{31}$$

This is nothing but BS equation for null interest rate. This equation admits of course an only regular solution $BS(t, s_1)$.

Let us state the comparison theorem, which is a key result of the viscosity theory (see Crandall et al., 1992) :

Theorem 2 *Let F be proper (Crandall et al., 1992), u an upper semi-continuous sub-solution of $F = 0$ and v a lower semi-continuous super-solution of $F = 0$ such that $u(T) \leq v(T)$. Then $u(t) \leq v(t)$ for all $t \leq T$.*

The comparison theorem implies, because of the super-solution property of V_* :

$$V_*(t, s_1) \geq BS(t, s_1). \tag{32}$$

We have just shown that in the case when the volatility could not jump, the super-replication price was bigger than the solution of the BS equation, which is rather intuitive.

4.2 Jumps of volatility

In this subsection, we consider the case when only the volatility can make jumps. Mathematically speaking, the price process of the negotiable asset is driven, under any of the equivalent probability measures, by the following SDE :

$$\frac{dS_t^1}{S_{t^-}^1} = \sigma_1(t, S_t^1, S_t^2)d\tilde{W}_t.$$

As for the process of the fictitious asset S_2, its dynamic remains the same. In the Bellman equation, we just have to set $\phi_1 = 0$. In fact, this condition kills the jump term $\mathcal{J}$ and the only remaining term leads to a well known viscosity equation for the function V_* :

$$-v_t + \inf_K \left[-\frac{1}{2} s_1^2 \sigma_1^2(t, s_1, s_2) v_{s_1 s_1} \right] \geq 0.$$

In fact, the presence of jumps does not modify the super price with respect to the case of continuous stochastic volatility, as it was shown by Frey (Frey, 1998). We recover here all the results obtained by Frey, as a particular case of our general model. The conclusion here, is that when the volatility is unbounded, the super-replication strategy is a trivial buy and hold strategy. When the volatility is bounded, we recover the same results as Avellaneda et al. (1995) for uncertain volatility model : the super price is a solution of a Black-Scholes-Barenblatt equation.

4.3 General case

We now turn to the general case when both the price and the volatility can jump. Actually, we have to distinguish the cases of unbounded and bounded volatility, because the mathematical treatment and the results turn out to be rather different. When the volatility is not bounded, the following theorem gives the same result as in the case of a continuous volatility, which is not surprising ; however, it is interesting to provide a proof.

Theorem 3 (Unbounded volatility) *We assume that the volatility is unbounded :*

$$\sup_{s_2} \sigma_1(t, s_1, s_2) = +\infty, \tag{33}$$

$$\inf_{s_2} \sigma_1(t, s_1, s_2) = 0 \tag{34}$$

Then, $V_*(t, s_1)$ is concave in s_1, decreasing in t and :

$$V_*(t, s_1) = g^{conc}(s_1).$$

where $g^{conc}(x)$ is the concave envelope of g. This corresponds to a buy-and-hold strategy.

Proof :

• We first show that V_* is concave in s_1. V_* is super-solution of $\inf_K[-\mathcal{L}v] = 0$. So, for each s_2, V_* is a super-solution of the equation $\mathcal{L}v = 0$. For instance, there always exists an element of K such that σ_1 is arbitrarily large. It means that we can choose an arbitrarily large volatility, and, at the limit, an infinite volatility ; as a consequence, V_* must be super-solution of :

$$-v_{s_1 s_1} = 0. \tag{35}$$

$V_*(t, \cdot)$ is also a super-solution of this equation, and, this is a known result from viscosity theory ; we conclude that V_* is concave in s_1.

• As above, it is possible to choose the volatility arbitrarily close to 0 and we prove exactly the same way that V_* is a viscosity super-solution of $-v_t = 0$. Another known result coming from the comparison theorem leads to the conclusion that V_* is a decreasing function of t.

• We now perform the calculation of the exact solution of $\inf_K[-\mathcal{L}v] = 0$. Here we need $g(x)/x$ bounded. By definition of the concave envelope of g, we have :

$$g^{conc}(s) = \inf\{c \in \mathbb{R} \ : \ \exists\, \Delta \in \mathbb{R}, \forall z > 0,\, c + \Delta(z - s) \geq g(z)\}.$$

From the boundary conditions and the comparison theorem, we get :

$$\forall (t, s_1) \in [0, T[\times \mathbb{R}_+^* \quad V_*(t, s_1) \geq g(s_1). \tag{36}$$

and so *a fortiori* :

$$\forall (t, s_1) \in [0, T[\times \mathbb{R}_+^* \quad V_*(t, s_1) \geq g^{conc}(s_1). \tag{37}$$

On the other hand, the above expression for g^{conc} proves the existence of a $\Delta > 0$ such that :

$$\forall s_1 > 0 \quad g^{conc}(s_1) + \Delta(S_T^1 - s_1) \geq g(S_T^1). \tag{38}$$

This means that when begining with an initial wealth $g^{conc}(s_1)$, and performing a *buy and hold* strategy (we buy Δ assets at $t = 0$, and we keep a static portfolio until T), we end up with a wealth bigger than $g(S_T^1)$. It means that the contingent claim has been over-hedged :

$$\forall (t, s_1) \in [0, T[\times \mathbb{R}_+^* \quad V_*(t, s_1) \leq g^{conc}(s_1).$$

Finally, $V_*(t, s_1) = g^{conc}(s_1)$ is a super-solution of the equation $\inf_K[-\mathcal{L}v] = 0$.

• The super-solution we have just found for $\inf_K[-\mathcal{L}v] = 0$ is concave in s_1, so that it is also a super-solution of $\inf_K[-\mathcal{J}v] = 0$. As a conclusion, we have shown that this was the exact solution to the general super price problem in jumping incomplete markets. The corresponding trading strategy is a *buy and hold* strategy.

□

In the case of bounded volatility, we are not able to solve the problem exactly ; nevertheless we can write down a non trivial upper limit to the super price. The following theorem extends the result of Avelleneda et al. (1995) to the case of jump processes.

Theorem 4 (Bounded volatility) *We make the assumption of a bounded volatility,* ie *:*

$$\sup_{s_2} \sigma_1(t, s_1, s_2) = \overline{\sigma}_1(t, s_1), \tag{39}$$

$$\inf_{s_2} \sigma_1(t, s_1, s_2) = \underline{\sigma}_1(t, s_1). \tag{40}$$

Then V_* is lower than the concave envelope of the only regular solution of the Black-Scholes-Barenblatt equation :

$$-v_t + \frac{1}{2}\underline{\sigma}_1^2 s_1^2 (v_{s_1 s_1})^- - \frac{1}{2}\overline{\sigma}_1^2 s_1^2 (v_{s_1 s_1})^+ = 0.$$

Proof :

The equation $\inf_K [-\mathcal{L}v] = 0$ is written in a more transparent way :

$$-v_t + \frac{1}{2}\underline{\sigma}_1^2 s_1^2 (v_{s_1 s_1})^- - \frac{1}{2}\overline{\sigma}_1^2 s_1^2 (v_{s_1 s_1})^+ = 0.$$

When this equation coefficients are assumed to be regular enough, there exists a unique regular solution called $\tilde{V}(t, s_1)$. The comparison theorem imposes :

$$\forall (t, s_1), \;\; V_*(t, s_1) \geq \tilde{V}(t, s_1).$$

On the other hand, the concave envelope of $\tilde{V}$ is also a super-solution of the Black-Scholes-Barenblatt equation, which satisfies the positivity constraint of the jump term ; then :

$$V_*(T^-, s) \leq g(s) \leq \tilde{V}^{conc}(T^-, s).$$

The comparison theorem gives :

$$\forall (t, s_1), \; V_*(t, s_1) \leq \tilde{V}^{conc}(t, s_1).$$

and this achieves the proof.

□

The super price, because of the comparison theorem, is a value between the concave envelope of the solution of the BS equations respectively associated to $\underline{\sigma}_1$ and $\overline{\sigma}_1$. We found here a very non trivial upper limit to this super-replication price.

The case of deterministic volatility is a particular case of bounded volatility, with $\sigma_1 = \underline{\sigma}_1 = \overline{\sigma}_1$. The function V_* is between the solution of the BS type equation with volatility σ_1 and its concave envelope.

Conclusion

The problem of super-replication in an incomplete market (for instance in the case of unbounded stochastic volatility and transaction costs) often leads to trivial arbitrage bounds for the prices of contingent claims. In this article we solve the case where the risky asset itself and/or its volatility can jump, and we make very few assumptions on the payoff function. We restrict the exploration of the equivalent martingale measures set to the subset of the markovian probability changes : the maximum of the expected cash flow of a contingent claim on this subset of equivalent martingale measures is called super price. We expect that the nature of the result remains unchanged since in all the known cases, the super price is equal to the super-replication price.

We show that the super price of a european contingent claim is the solution of a non linear integro-differential equation. We solve explicitely this equation when this is possible. For instance, when the volatility only can jump and is bounded, we recover Avellaneda's uncertain volatility model : in this special case, the equation to be solved is Black-Scholes-

Barenblatt's equation. In one case (both the underlying asset and the bounded volatility can jump), we have not been able to find an explicit analytical solution for the super price equation, but we have found non trivial bounds for the super price.

A Jump processes

This appendix is devoted the generalization of Itô's calculus when Poisson processes are also present.

Let us consider a 2-dimensional process $(S_t^1, S_t^2)_{t\geq 0}$ satisfying the following SDE :

$$\begin{cases} \frac{dS_t^1}{S_{t-}^1} = \mu_1\, dt + \sigma_1\, dW_t + \phi_1\, dN_t \\ \frac{dS_t^2}{S_{t-}^2} = \mu_2\, dt + \sigma_2\, dW_t + \phi_2\, dN_t \end{cases}.$$

$(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, which is an increasing process. This property is important because we easily define stochastic integrals with respect to a Poisson process *à la Stieltjes* for each $\omega \in \Omega$:

$$\int_0^T f(t)dN_t = \sum_{T_i \leq T} f(T_i)$$

where the T_i are the random dates (stopping times) at which the Poisson process jumps. We put $Y_t = f(t, S_t^1, S_t^2)$; so, the extention of Itô's formula gives :

$$\begin{aligned} dY_t &= \left(f_t + \mu_1 S_t^1 f_{S^1} + \mu_2 S_t^2 f_{S^2}\right) dt + \left(\sigma_1 S_t^1 f_{S^1} + \sigma_2 S_t^2 f_{S^2}\right) dW_t + \delta f\, dM_t && (41) \\ &+ \frac{1}{2}\left(\sigma_1^2 (S_t^1)^2 f_{S^1S^1} + 2\sigma_1\sigma_2 S_t^1 S_t^2 f_{S^1S^2} + \sigma_2^2 (S_t^2)^2 f_{S^2S^2}\right) dt && (42) \\ &+(\delta f - \phi_1 S_{t^-}^1 f_{S^1} - \phi_2 S_{t^-}^2 f_{S^2})\lambda dt, && (43) \end{aligned}$$

where $\delta f = f(t, S_{t^-}^1(1+\phi_1), S_{t^-}^2(1+\phi_2)) - f(t, S_{t^-}^1, S_{t^-}^2)$

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