

Simulation approaches to risk management

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Computers and computer science have become widely used in the financial industry and they offer to risk managers and R&D analysts a powerful tool for designing and calibrating their models. In particular, when analytic calculations cannot be handled, the computer offers the quantitative analyst the ability of simulating the phenomenon he wants to study. The Monte-Carlo approach is then extensively used because it is at the same time easy to set up and very flexible.

In this article, we first explain how simulations are used to compute mathematical indicators. Second, we give some details on how to generate random variables. Third, we explain how to increase the accuracy of Monte-Carlo estimates. Finally, we give examples of Monte-Carlo techniques in the field of finance.

Monte-Carlo methods are based on the link between probabilities and volumes in the probability space. Whereas the mathematical theory of measure expresses the probability of an event in terms of the volume of this event in the universe of possible outcomes, Monte-Carlo methods do exactly the reverse by estimating the volume of an event and interpreting it as a probability. In the basic Monte-Carlo method, we sample randomly a given number of outcomes from the universe of possible outcomes, and the fraction of the outcomes that fall in a given set is an estimate of the set's volume. The law of large numbers ensures the convergence of this estimate to the real value when the number of draws goes to infinity. To make this more concrete, we consider a random experiment that has several possible outcomes (for instance tossing a coin has two possible outcomes). In our Monte-Carlo simulation, we make N tosses, and we assume that outcome number i occurs N_i times. If outcome number i has a probability of occurrence equal to p_i , then we have:

$$\frac{N_i}{N} \xrightarrow{N \rightarrow \infty} p_i \quad (1)$$

We consider now a continuous real random variable X with density function f , and we seek to compute the expected value of the random variable $g(X)$. We assume that the $(X_i)_{1 \leq i \leq N}$ are a set of N outcomes of the random variable X . The Monte-Carlo technique comes from the following relationship:

$$\frac{1}{N} \sum_{i=1}^N g(X_i) \xrightarrow{N \rightarrow \infty} E[g(X)] \quad (2)$$

As shown by the law of large numbers, this convergence is unfortunately very slow, and the standard deviation of the estimate of the expectation decreases only as $1/\sqrt{N}$. Dividing the estimate error by a factor 10 requires a sample 100 times larger. However, the universality of the law of large numbers makes Monte-Carlo techniques relevant to any situation, whatever its complexity, but also holds its weakness because of time consumption of accurate estimates.

As we see in this simple example, we can estimate mathematical indicators by just simulating random variables. The accuracy of the result will not only come from the size of the random sample we use, but also from the random variables generator. The key ingredient is the random number generator that samples sequences of uniformly distributed random numbers on the interval $[0, 1]$. This algorithm is likely to be used millions of times in a Monte-Carlo simulation and requires an efficient implementation. The second chapter of Glassermann's book (Glassermann, 2004) provides with a clear introduction and references to this topic. Once a uniform random variable U is generated, there are several methods to generate a random variable X with density function f :

1. Inverse transform method: Let's call F the cumulative distribution function of the random variable X . The inverse transform method sets: $X = F^{-1}(U)$. This method is the easiest one when we are able to express analytically the inverse function of F .
2. Acceptance-rejection method: this method is useful when the density is defined on a bounded interval and remains bounded, but the cumulative distribution function cannot be express in terms of simple

functions. The typical example is the beta function. Let m such that for any value of x , we have $f(x) < m$. We sample the uniform random variable U , and another uniformly distributed random variable V on the interval $[0, m]$. If $V < f(U)$, we keep U and we set $X=U$; on the reverse case, we reject U and we restart.

3. Normal random variables: they are building blocks of many financial applications and require a specific methodology. There are several ways of doing this. A first one is to use approximations of either the inverse normal function of the cumulative normal distribution. A second one is to generate two uniform random variables U_1 and U_2 and to apply the Box-Muller formula $X = \sqrt{-2 \ln U_1} \cos(U_2)$

In financial applications, random variable generation is generally the first step of the Monte-Carlo simulation. As the real financial world and the managers' strategies are essentially dynamic, the second step is to generate path samples. The simulations become rapidly complex because of the dynamics of the phenomenon. The models often involve stochastic differential equations that need to be discretized in time for numerical purposes. Time discretization is another source of biases in the Monte-Carlo estimates, especially when threshold effects are involved. We refer to Glassermann's book (Glassermann, 2004), chapter 6, for an extensive discussion.

The major disadvantage with Monte-Carlo simulations is time consumption. The convergence of the statistical indicators is very slow and, generally, Monte-Carlo simulations are used for very complex systems and require a lot of computing. Variance reduction techniques are extensions of the basic Monte-Carlo technique and lead to a significant improvement of the convergence. The three main techniques are the antithetic variates, the control variates, and importance sampling. As we shall see, each technique is relevant in a given situation.

The first technique is antithetic variates. It is useful when the probability density of the random variable (or random path) to be generated is a symmetric function. This is for instance the case for the normal variable or the Gaussian processes. The trick is to generate an outcome X_i of the random variable X , and to note that $-X_i$ has exactly the same probability density. This allows at the same time to double size the set of generated outcomes without increasing the time consumption and to symmetrize the outcomes. Variance reduction is then significant, and the Monte Carlo expectation writes:

$$\frac{1}{2N} \sum_{i=1}^N [g(X_i) + g(-X_i)] \xrightarrow{N \rightarrow \infty} E[g(X)] \quad (3)$$

The second technique is the control variates. When running a Monte-Carlo simulation, the error we make is unknown except when the indicator of interest is already known. This technique uses information about the error in an estimate of a known quantity to correct the estimate of an unknown quantity. Of course, the method performs well when the known and unknown quantities are close to each other. In this case, we make the assumption that the error made on the known quantity is equal to the error made on the unknown quantity and then, we improve the accuracy of the estimate.

The third method is importance sampling. This method consists in changing the probability measure in order to generate the sample set in the range of values that contribute maximally to the quantity of interest. The method is a standard trick in probability theory and is based on the following relationship:

$$E[g(X)] = \int g(x) f(x) dx = \int g(x) \frac{f(x)}{h(x)} h(x) dx = \tilde{E} \left[g(X) \frac{f(X)}{h(X)} \right] \quad (4)$$

where the second expectation is computed under the density measure h instead of f , and the likelihood ratio compensates the change of measure. This technique is interesting when the outcomes that contribute most to the expectation are located far in the tails of the distribution function of the random variable and are very rare events. The new measure makes these rare outcomes more probable and must be chosen so that the variance of the estimation under the new measure is smaller than under the original measure.

Other methods such as stratified sampling and moments matching are useful to increase the performance of Monte-Carlo simulations. We also emphasize on the possibility to combine several methods. For instance for risk management purposes, we often use both important sampling and control variates in order to design the extreme losses distribution of a financial portfolio. The reader can find extensive details in the references given below.

The applications of Monte-Carlo simulations in financial engineering are mainly oriented towards the pricing and hedging of derivatives and the risk management. In the context of derivatives, the complexity of the products is increasing everyday and requires more and more numerical expedients. On the risk management side, simulations are often used to compute the profit and loss distribution function of a portfolio, and they help in

focusing on the extreme losses. Other applications are in between the two previous ones; among others, we have portfolio optimization, Asset-Liabilities Management, insurance premiums pricing, and many applications to the decision process.

Further reading

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